Higher Mathematics

Vectors - **Solutions** - 2013-2019

Marks are indicated in brackets after each question number

2013 Paper 1 Question 12, (2)

$$\underline{f} + \underline{g} = \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix}$$

$$\left|\underline{f} + \underline{g}\right| = \sqrt{5^2 + 4^2 + 5^2}$$
$$= \sqrt{64}$$
$$= 8$$

Question 13

Let $x^2 - 7x + 12 = 0$

$(x-4)(x-3) = 0$

$x = 3, x = 4$

So $x = 3, x = 4$ cannot be in the domain of $f(x)$

2013 Paper 1 Question 14, (2)

$$\underline{a} \,.\, (\underline{a} + \underline{b}) = \underline{a} \,.\, \underline{a} + \underline{a} \,.\, \underline{b}$$

$= \left|\underline{a}\right|\left|\underline{a}\right| cos\theta + 5$ where θ is the angle between $\underline{a}$ and itself

$$= 3 \bullet 3 \bullet \cos 0 + 5$$
$$= 9 + 5$$
$$= 14$$

2013 Paper 1 Question 24, (4) (5)

a) i) $\overrightarrow{AT} = \begin{pmatrix} 10 \\ 10 \\ 4 \end{pmatrix} \quad \overrightarrow{TB} = \begin{pmatrix} 15 \\ 15 \\ 6 \end{pmatrix}$

$$\overrightarrow{AT} = 2\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} \quad \overrightarrow{TB} = 3\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$$

$$\frac{1}{2}\overrightarrow{AT} = \frac{1}{3}\overrightarrow{TB}$$

$$3\overrightarrow{AT} = 2\overrightarrow{TB}$$

This shows that $\overrightarrow{AT}$ & $\overrightarrow{TB}$ are parallel but since T is a common point,

It follows that A, T, & B are collinear.

ii) The ratio is 2 : 3

b) Since C lies on the x-axis it has a y co-ordinate of 0 and a z co-ordinate of 0.

Let $C = (c,\ 0, 0)$

Then $\overrightarrow{TC} = C - T = \begin{pmatrix} c-3 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{TB} \bullet \overrightarrow{TC} = 0$ since they are perpendicular

$$\overrightarrow{TB} \bullet \overrightarrow{TC} = 15(c-3) + \left(15 \bullet 0\right) + \left(6 \bullet 0\right) = 0$$

$$15c - 45 = 0$$

$$c = 3$$

2014 Paper 1 Question 6, (2)

$$2\underline{u} - 3\underline{v} = 2\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ 6 \end{pmatrix}$$

2014 Paper 1 Question 14, (2)

Since $\underline{u}$ & $\underline{v}$ are perpendicular $\underline{u} \bullet \underline{v} = 0$

$$\underline{u} \bullet \underline{v} = (1 \bullet -6) + 2k + 5k = 0$$

$$7k - 6 = 0$$

$$k = \frac{6}{7}$$

2014 Paper 1 Question 16, (2)

$$\underline{a} \bullet (\underline{a} + 2\underline{b}) = \underline{a} \bullet \underline{a} + 2\underline{a} \bullet \underline{b}$$

$= |\underline{a}||\underline{a}|cos\theta + 2 \bullet \frac{2}{3}$ where θ is the angle between $\underline{a}$ and itself

$$= 1 \bullet 1 \bullet 1 + \frac{4}{3}$$

$$= \frac{7}{3}$$

2014 Paper 1 Question 19, (2)

Let C be the centre of the hexagon

$$\overrightarrow{SW} = \overrightarrow{SR} + \overrightarrow{RC} + \overrightarrow{CW}$$

$$= -\underline{u} - \underline{v} - \underline{v}$$

$$= -\underline{u} - 2\underline{v}$$

2014 Paper 2 Question 4, (2) (2) (5)

a) $C = \left(11, 12, 6\right),\ D = (8, 8, 4)$

b) $\vec{CB} = \begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix} \quad \vec{CD} = \begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix}$

c) $\left|\vec{CB}\right| = \sqrt{0^2 + (-8)^2 + (-4)^2} = \sqrt{80}$

$\left|\vec{CD}\right| = \sqrt{(-3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}$

$\vec{CB} \bullet \vec{CD} = (0 \bullet -3) + (-8 \bullet -4) + (-4 \bullet -2) = 40$

$\vec{CB} \bullet \vec{CD} = \left|\vec{CB}\right|\left|\vec{CD}\right|\cos(BCD)^\circ$

$40 = \sqrt{80}\sqrt{29}\cos(BCD)^\circ$

$\frac{40}{\sqrt{80}\sqrt{29}} = \cos(BCD)^\circ$

$BCD = cos^{-1}(\frac{40}{\sqrt{80}\sqrt{29}})$

$BCD = 33.85^\circ$

2015 Paper 1 Question 1, (2)

Since $\underline{u}$ & $\underline{v}$ are perpendicular $\underline{u} \bullet \underline{v} = 0$

$\underline{u} \bullet \underline{v} = \left(8 \bullet (-3)\right) + (2 \bullet t) + \left((-1) \bullet \left(-6\right)\right) = 0$

$-24 + 2t + 6 = 0$

$2t = 18$

$t = 9$

2015 Paper 2 Question 6, (3) (1) (3)

a) $\underline{p} \bullet \left(\underline{q} + \underline{r}\right) = \underline{p} \bullet \underline{q} + \underline{p} \bullet \underline{r}$

$$= \left(\left|\underline{p}\right|\left|\underline{q}\right|\cos 60°\right) + (\left|\underline{p}\right|\left|\underline{r}\right|\cos 90°)$$

$$= \left(3 \bullet 3 \bullet \frac{1}{2}\right) + 0 = \frac{9}{2}$$

b) $\overrightarrow{EC} = -\underline{q} + \underline{p} + \underline{r}$

c) $\overrightarrow{AE} \bullet \overrightarrow{EC} = \underline{q} \bullet (-\underline{q} + \underline{p} + \underline{r})$

$$= -\underline{q} \bullet \underline{q} + \underline{q} \bullet \underline{p} + \underline{q} \bullet \underline{r}$$

$$= -\left|\underline{q}\right|\left|\underline{q}\right|\cos 0 + \frac{9}{2} + \left|\underline{q}\right|\left|\underline{r}\right|\cos 30°$$

[to see where the $30°$ comes from consider the diagram carefully)

$$= -9 + \frac{9}{2} + \frac{3\sqrt{3}}{2}\left|\underline{r}\right|$$

But since $\overrightarrow{AE} \bullet \overrightarrow{EC} = 9\sqrt{3} - \frac{9}{2}$ we have

$$9\sqrt{3} - \frac{9}{2} = -9 + \frac{9}{2} + \frac{3\sqrt{3}}{2}\left|\underline{r}\right|$$

Multiplying through by 2 gives

$$18\sqrt{3} - 9 = -18 + 9 + 3\sqrt{3}\left|\underline{r}\right|$$

$$18\sqrt{3} = 3\sqrt{3}\left|\underline{r}\right|$$

$$\left|\underline{r}\right| = 6$$

2016 Paper 1 Question 7, (2) (2)

a) $\overrightarrow{FH} = \overrightarrow{FG} + \overrightarrow{GH} = \underline{i} + 3\underline{j} - 4\underline{k}$

b) $\overrightarrow{FE} + \overrightarrow{FH} + \overrightarrow{HE} = \overrightarrow{FH} - \overrightarrow{EH} = -\underline{i} - 5\underline{k}$

2016 Paper 1 Question 11, (2) (3)

a) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

Since the ratio is $1:2$ there are 3 'parts' to the division

$$\frac{1}{3}\overrightarrow{AC} = \frac{1}{3}\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

So, $B = (1+1, 3-2, -2+2) = (2, 1, 0)$

b) $k\overrightarrow{AC} = \begin{pmatrix} 3k \\ -6k \\ 6k \end{pmatrix}$

$$\begin{aligned} \left|k\overrightarrow{AC}\right| &= \sqrt{(3k)^2 + (-6k)^2 + (6k)^2} \\ &= \sqrt{9k^2 + 36k^2 + 36k^2} \\ &= \sqrt{81k^2} \\ &= 9k \end{aligned}$$

Since $\left|k\overrightarrow{AC}\right| = 1$ we have

$$9k = 1$$

$$k = 1/9$$

2016 Paper 2 Question 5, (2) (4)

a) $\overrightarrow{AB} = \begin{pmatrix} -8 \\ -16 \\ -2 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} -2 \\ -8 \\ -16 \end{pmatrix}$

b) $\overrightarrow{AB} \bullet \overrightarrow{AC} = (-8 \bullet -2) + \left(-16 \bullet -8\right) + (-2 \bullet -16)$

$= 112$

$\left|\overrightarrow{AB}\right| = \sqrt{(-8)^2 + (-16)^2 + (-2)^2} = 18$

$\left|\overrightarrow{AC}\right| = \sqrt{(-2)^2 + (-8)^2 + (-16)^2} = 18$

Let θ be the angle BAC

$\overrightarrow{AB} \bullet \overrightarrow{AC} = \left|\overrightarrow{AB}\right| \left|\overrightarrow{AC}\right| \cos\theta$

$112 = 18 \bullet 18\cos\theta$

$112 = 324\cos\theta$

$\frac{112}{324} = \cos\theta$

$\theta = cos^{-1}\left(\frac{112}{324}\right) = 69.8\ ^\circ$

2017 Paper 1 Question 5, (1) (3)

a) $\underline{u} \bullet \underline{v} = \left(5 \bullet 3\right) + (1 \bullet -8) + \left(-1 \bullet 6\right) = 1$

b) $\left|\underline{u}\right| = \sqrt{5^2 + 1^2 + (-1)^2} = \sqrt{27}$

$\underline{u} \bullet \underline{w} = \left|\underline{u}\right| \left|\underline{w}\right| \cos(\frac{\pi}{3})$

$= \sqrt{27} \bullet \sqrt{3} \bullet \frac{1}{2}$

$= \sqrt{81} \bullet \frac{1}{2} = \frac{9}{2}$

2017 Paper 2 Question 5, (2) (2) (5)

a) $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ} = \left(9\underline{i} + 5\underline{j} + 2\underline{k}\right) + \left(-12\underline{i} - 9\underline{j} + 3\underline{k}\right) = -3\underline{i} - 4\underline{j} + 5\underline{k}$

b) $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$

$$= \overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QR}$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + \frac{1}{3}(-\overrightarrow{RQ})$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + \frac{1}{3}(12\underline{i} + 9\underline{j} - 3\underline{k})$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + 4\underline{i} + 3\underline{j} - \underline{k}$$

$$= \underline{i} - \underline{j} + 4\underline{k}$$

c) $\overrightarrow{PQ} \bullet \overrightarrow{PS} = -3 + 4 + 20 = 21$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-3)^2 + (-4)^2 + 5^2} = \sqrt{50}$$

$$\left|\overrightarrow{PS}\right| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18}$$

Let θ be the angle between $\overrightarrow{PS}$ and $\overrightarrow{PQ} = QPS$

$$21 = \sqrt{50} \bullet \sqrt{18} \bullet \cos\theta$$

$$\frac{21}{\sqrt{50}\sqrt{18}} = \cos\theta$$

$$\theta = cos^{-1}\left(\frac{21}{\sqrt{50}\sqrt{18}}\right)$$

$$\theta = 45.6°$$

2018 Paper 1 Question 9, (1) (2)

a) $\overrightarrow{BC} = -\underline{t} + \underline{u} = \underline{u} - \underline{t}$

b) $\overrightarrow{MD} = \overrightarrow{MC} + \overrightarrow{CA} + \overrightarrow{AD}$

$$= \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AD}$$

$$= \frac{1}{2}\left(\underline{u} - \underline{t}\right) - \underline{u} + \underline{v}$$

$$= \frac{1}{2}\underline{u} - \frac{1}{2}\underline{v} - \underline{u} + \underline{v}$$

$$= \underline{v} - \frac{1}{2}\underline{u} - \frac{1}{2}\underline{t}$$

2018 Paper 1 Question 12, (1) (3)

a) $\underline{a} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -2 \\ 1 \\ p \end{pmatrix}$

$$2\underline{a} + \underline{b} = 2\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ p \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ p \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \\ 4 + p \end{pmatrix}$$

b) $\left|2\underline{a} + \underline{b}\right| = \sqrt{6^2 + (-3)^2 + (4+p)^2} = 7$

$$36 + 9 + 16 + 8p + p^2 = 49$$

$$p^2 + 8p + 12 = 0$$

$$(p+6)(p+2) = 0$$

$$p = -6, p = -2$$

2018 Paper 2 Question 2, (1) (4)

a) $\underline{u} \bullet \underline{v} = 7 + 32 - 15 = 24$

b) $\left|\underline{u}\right| = \sqrt{1+16+9} = \sqrt{26}$

$\left|\underline{v}\right| = \sqrt{49+64+25} = \sqrt{138}$

$$\cos\theta = \frac{24}{\sqrt{26}\sqrt{138}}$$

$$\theta = cos^{-1}(\frac{24}{\sqrt{26}\sqrt{138}})$$

$$= 66.38°$$

2019 Paper 1 Question 5, (3) (1)

a) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$4\vec{AB} = 3\vec{BC}$ so $\vec{AB}$ and $\vec{BC}$ are parallel but since B is a common point, A, B, & C are collinear.

b) 3 : 4

<u>2019 Paper 1 Question 9, (1) (3) (2)</u>

a) i) $\underline{u} \bullet \underline{v} = p(2p + 16) + (-2 \bullet -3) + (4 \bullet 6)$

$= 2p^2 + 16p + 30$

ii) For perpendicular vectors $\underline{u} \bullet \underline{v} = 0$, giving

$2p^2 + 16p + 30 = 0$

$p^2 + 8p + 15 = 0$

$(p + 5)(p + 3) = 0$

$p = -5, p = 3$

b) If $\underline{u}$ and $\underline{v}$ are parallel, there exists a number k such that $\underline{u} = k\underline{v}$ i.e. one vector is a multiple of the other.

Comparing the y component of $\underline{u}$ and $\underline{v}$ we have $-2 = k(-3)$, giving $k = \frac{2}{3}$

So, $p = \frac{2}{3}(2p + 16)$

$3p = 4p + 32$

$p = -32$

2019 Paper 2 Question 3, (1) (2)

a) $\overrightarrow{BE} = -\underline{p} + \underline{r}$

b) $\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF}$

$$= -\underline{r} + \underline{p} + \frac{3}{4}\underline{q}$$

2019 Paper 2 Question 14, (4)

Let θ be the angle between $\underline{u}$ $\underline{v}$

$$\underline{u} \bullet \left(\underline{u} + \underline{v}\right) = \underline{u} \bullet \underline{u} + \underline{u} \bullet \underline{v}$$

$$= \left|\underline{u}\right|\left|\underline{u}\right|\cos 0 + \left|\underline{v}\right|\left|\underline{v}\right|\cos\theta$$

$$= (4\ x\ 4\ x\ 1) + (4\ x\ 5\ x\cos\theta)$$

$$= 16 + 20\cos\theta$$

Since $\underline{u} \bullet \left(\underline{u} + \underline{v}\right) = 21$ we have

$$16 + 20\cos\theta = 21$$

$$20\cos\theta = 5$$

$$\cos\theta = \frac{5}{20}$$

$$\theta = cos^{-1}(\frac{5}{20})$$

$$\theta = 75.5°$$