Higher Mathematics

Trigonometric Identities - **Solutions** - 2013-2019

Marks are indicated in brackets after each question number

2013 Paper 1 Question 10, (2)

$$\cos(270 - a) = \cos 270\cos a + \sin 270\sin a$$
$$= 0 \bullet \cos a + (-1) \bullet \sin a$$
$$= -\sin a$$

2013 Paper 1 Question 23, (4) (2)

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x - a) = k\sin x\cos a - k\cos x\sin a$

$$= k\cos a\sin x - k\sin a\cos x$$

By inspection we have $\sqrt{3} = k\cos a$ and $1 = k\sin a$

$$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\frac{k\sin a}{k\cos a} = \frac{1}{\sqrt{3}} = \tan a$$

Giving $a = 30°$ by exact values

Since we want a to be such that both $\sin a$ and $\cos a$ are positive, by CAST,

a must be in quadrant 1. So, $30°$ is correct

Thus $\sqrt{3}\sin x - \cos x = 2\sin(x - 30)°$

b) $4 + 5\cos x - 5\sqrt{3}\sin x = 4 - 5(\sqrt{3}\sin x - \cos x)$

$= 4 - 5 \bullet 2\sin(x - 30)$ from a)

$= 4 - 10\sin(x - 30)$

$$= -10\sin(x - 30) + 4$$

$-10\sin(x - 30)$ has a maximum value of 10

So, $= -10\sin(x - 30) + 4$ has a maximum value of 14

2014 Paper 1 Question 4, (2)

Let $3\sin x - 4\cos x = k\cos(x - a)$

$$= k\cos x\cos a + k\sin x\sin a$$

$$= k\cos a\cos x + k\sin a\sin x$$

By inspection $3 = k\sin a$ and $-4 = k\cos a$

2014 Paper 1 Question 7, (2)

$\sin 2a = 2\sin a\cos a$

From SOHCAHTOA

$$\sin 2a = 2 \bullet \frac{3}{\sqrt{34}} \bullet \frac{5}{\sqrt{34}}$$

$$= \frac{30}{34} = \frac{15}{17}$$

2014 Paper 1 Question 18, (2)

$1 - 2\sin^2 15° = \cos(2 \bullet 15°)$ from the double angle formulas

$$= \cos 30°$$

$$= \frac{\sqrt{3}}{2}$$

2015 Paper 1 Question 10, (1) (2)

a) Using right-triangle with sides, 3,4, & 5 we have

$$\cos 2x = \frac{4}{5}$$

b) $\cos 2x = 2\cos^2 x - 1$

$$\frac{4}{5} = 2(\cos x)^2 - 1$$

$$\frac{9}{5} = 2(\cos x)^2$$

$$\frac{9}{10} = (\cos x)^2$$

$$\cos x = \sqrt{\frac{9}{10}}$$

<u>2015 Paper 2 Question 9, (8)</u>

Let $36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) = k\sin(1.5t - a)$

$$= k\sin\left(1.5t\right)\cos a - k\cos\left(1.5t\right)\sin a$$

$$= k\cos a\sin\left(1.5t\right) - k\sin a\cos(1.5t)$$

By inspection we have $36 = k\cos a$ and $15 = k\sin a$

$k = \sqrt{36^2 + 15^2} = 39$

$$\frac{k\sin a}{k\cos a} = \frac{15}{36} = \tan a$$

$a = tan^{-1}\left(\frac{15}{36}\right) = 0.39$ radians (confirm this answer is in the correct quadrant using CAST)

So, $36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) = 39\sin(1.5t - 0.39)$

$h = 36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) + 65$

$= 39\sin(1.5t - 0.39) + 65$ (from above)

Let $h = 100$ to give

$100 = 39\sin(1.5t - 0.39) + 65$

$35 = 39\sin(1.5t - 0.39)$

$$\frac{35}{39} = \sin(1.5t - 0.39)$$

$1.5t - 0.39 = \sin^{-1}\left(\frac{35}{39}\right) = 1.11$ radians

$1.5t = 1.5,\ t = 1$

Second solution for $\frac{35}{39} = \sin(1.5t - 0.39)$ is given by

$\pi - 1.11 = 2.03$ radians, so

$1.5t - 0.39 = 2.03$

$t = 1.61$

So, the solutions are $t = 1$ second and $t = 1.61$ seconds

2016 Paper 1 Question 13, (5)

$$\cos(q - p) = \cos p \cos q + \sin q \sin p$$

Use Pythagoras to work out the triangle hypotenuse

$$\cos(q - p) = \left(\frac{4}{5} \bullet \frac{4}{\sqrt{17}}\right) + (\frac{3}{5} \bullet \frac{1}{\sqrt{17}})$$

$$= \frac{16}{5\sqrt{17}} + \frac{3}{5\sqrt{17}}$$

$$= \frac{19}{5\sqrt{17}}$$

Multiply by $\frac{\sqrt{17}}{\sqrt{17}}$ to give

$$\cos(q - p) = \frac{19\sqrt{17}}{85}$$

2016 Paper 2 Question 8, (4) (4)

a) Let $5\cos x - 2\sin x = k\cos(x + a)$

$= k\cos x\cos a - k\sin a\sin x$

$= k\cos a\cos x - k\sin a\sin x$

By inspection $k\cos a = 5$ and $k\sin a = 2$

$k = \sqrt{5^2 + 2^2} = \sqrt{29}$

$\frac{k\sin a}{k\cos a} = \frac{2}{5} = \tan a$

$a = tan^{-1}\left(\frac{2}{5}\right) = 0.38$ radians

So, $5\cos x - 2\sin x = \sqrt{29}\cos(x + 0.38)$

b) Equating the line and curve gives

$10 + 5\cos x - 2\sin x = 12$

$5\cos x - 2\sin x = 2$

Using the result from part a) we have

$\sqrt{29}\cos(x + 0.38) = 2$

$\cos(x + 0.38) = \frac{2}{\sqrt{29}}$

First solution

$x + 0.38 = cos^{-1}(\frac{2}{\sqrt{29}})$

$x + 0.38 = 1.19$

$x = 0.81$ radians

Second solution

$x + 0.38 = 2\pi - 0.81$

$x = 4.71$ radians

2016 Paper 2 Question 11, (4) (2)

a) $\sin 2x \tan x = 2\sin x \cos x \tan x$

$= 2\sin x \cos x \dfrac{\sin x}{\cos x}$

$= 2\sin^2 x$

$= 1 - 2\cos 2x$ from double angle formula

b) $f(x) = \sin 2x \tan x$

$= 1 - 2\cos 2x$

$f'(x) = 4\sin 2x$

2017 Paper 1 Question 14, (4) (3)

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x - a)$

$= k\sin x \cos a - k\cos x \sin a$

$= k\cos a \sin x - k\sin a \cos x$

By inspection $\sqrt{3} = k\cos a$ and $1 = k\sin a$

$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

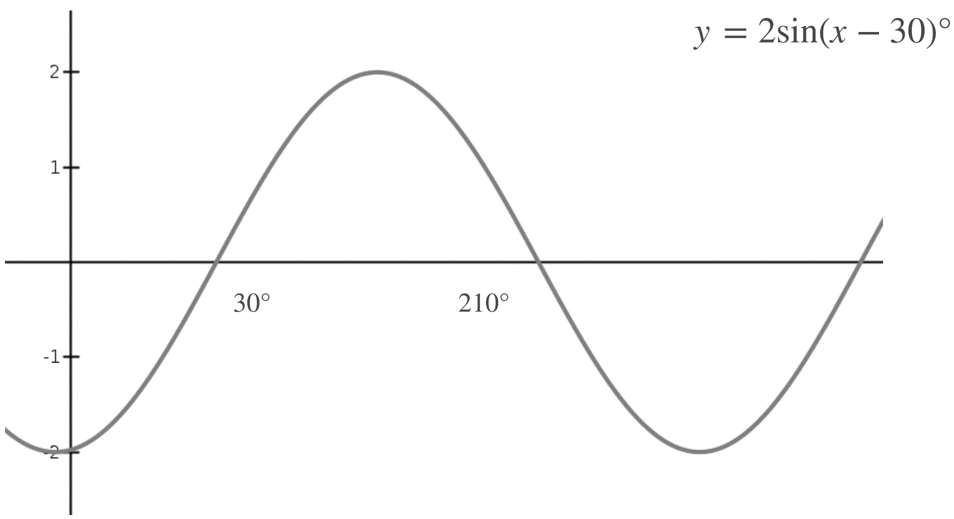
$\dfrac{k\sin a}{k\cos a} = \dfrac{1}{\sqrt{3}} = \tan a$

$a = tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30°$

Confirm that this answer is in the correct quadrant using CAST

So, $\sqrt{3}\sin x - \cos x = 2\sin(x - 30)°$

b) Since $\sqrt{3}\sin x - \cos x = 2\sin(x-30)°$ the graph of $y = \sqrt{3}\sin x - \cos x$ is the same as the graph of $y = 2\sin(x-30)$



2017 Paper 2 Question 11, (3) (3)

a) $\dfrac{\sin 2x}{2\cos x} - \sin x\cos^2 x = \dfrac{2\sin x\cos x}{2\cos x} - \sin x\cos^2 x$

$= \sin x - \sin x\cos^2 x$

$= \sin x - \sin x\,(1-\sin^2 x)$ since $\sin^2 x + \cos^2 x = 1$

$= \sin x - \sin x + \sin^3 x$

$= \sin^3 x$

b) $\dfrac{d}{dx}\left(\dfrac{\sin 2x}{2\cos x} - \sin x\cos^2 x\right) = \dfrac{d}{dx}\left(\sin^3 x\right)$

$= \dfrac{d}{dx}(\sin x)^3$

$= 3(\sin x)^2 \bullet \cos x$

$= 3\cos x\sin^2 x$

2018 Paper 1 Question 13, (3) (1) (3)

a) i) $\sin(2x) = 2\sin x\cos x$

$$= 2 \, x \, \frac{2}{\sqrt{11}} \, x \, \frac{\sqrt{7}}{\sqrt{11}}$$

$$= \frac{4\sqrt{7}}{11}$$

ii) $\cos(2x) = 1 - 2sin^2x$

$$= 1 - 2(\frac{2}{\sqrt{11}})^2$$

$$= 1 - 2(\frac{4}{11})$$

$$= 1 - \frac{8}{11}$$

$$= \frac{3}{11}$$

b) $\sin(3x) = \sin(2x + x)$

$$= \sin(2x)\cos x + \cos(2x)\sin x$$

$$= \left(\frac{4\sqrt{7}}{11} \, x \, \frac{\sqrt{7}}{11}\right) + (\frac{3}{11} \, x \, \frac{2}{11})$$

$$= \frac{28}{11\sqrt{11}} + \frac{6}{11\sqrt{11}}$$

$$= \frac{34}{11\sqrt{11}}$$

2018 Paper 2 Question 8, (4) (1) (2)

a) $2\cos x - \sin x = k\cos(x - a)$

$$= k\,cos\,x\cos a + k\sin x\sin a$$

$$= k\cos a\cos x + k\sin a\sin x$$

$2 = k\cos a$

$-1 = k\sin a$

$k = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$\tan a = -\frac{1}{2}$

$tan^{-1}\left(\frac{1}{2}\right) = 26.4°$

From CAST $a = 360 - 26.4 = 333.6°$

$2\cos x - \sin x = \sqrt{5}\cos(x - 333.6)°$

b) i) $6\cos x - 3\sin x = 3(2\cos x - \sin x)$

$$= 3\sqrt{5}\cos(x - 333.6)°$$

Considering the graph of this function gives a minimum value of $-3\sqrt{5}$

ii) $3\sqrt{5}\cos(x-333.6)^\circ = -3\sqrt{5}$

$\cos(x-333.6)^\circ = -1$

$x - 333.6 = 180$

$x = 180 + 333.6 = 513.4$

$x = 513.4 - 360 = 153.4^\circ$

<u>2019 Paper 1 Question 13, (1) (1) (3)</u>

a) i) $AC = \sqrt{(\sqrt{5})^2 - 1^2} = 2$

$\cos p = \dfrac{2}{\sqrt{5}}$

ii) $AD = 2 + 1 = 3$

$\cos q = \dfrac{3}{\sqrt{10}}$

b) $\sin\left(p+q\right) = \sin p \cos q + \cos p \sin q$

$= \left(\dfrac{1}{\sqrt{5}}\right)\left(\dfrac{3}{\sqrt{10}}\right) + (\dfrac{2}{\sqrt{5}})(\dfrac{1}{\sqrt{10}})$

$= \dfrac{3}{\sqrt{50}} + \dfrac{2}{\sqrt{50}}$

$= \dfrac{5}{\sqrt{50}}$

$= \dfrac{5}{5\sqrt{2}}$

$= \frac{\sqrt{2}}{2}$

<u>2019 Paper 1 Question 17, (3) (2)</u>

a) $(\sin x - \cos x)^2 = sin^2x + cos^2x - 2\sin x\cos x$

$= 1 - 2\sin x\cos x$ since $sin^2x + cos^2x = 1$

$= 1 - \sin 2x$

b) $\int (\sin x - \cos x)^2 dx = \int (1 - \sin 2x)\, dx$ using part a)

$= x + \frac{1}{2}\cos 2x + c$

<u>2019 Paper 2 Question 6, (4) (3)</u>

a) $2\cos x - 3\sin x = k\cos(x + a)$

$= k\cos x\cos a - k\sin x\sin a$

$= k\cos a\cos x - k\sin a\sin x$

$2 = k\cos a$ $3 = k\sin a$

$k = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\frac{3}{2} = \tan a$

$a = tan^{-1}\left(\frac{3}{2}\right) = 56.3°$ Check that this is in the correct quadrant using CAST

So, $2\cos x - 3\sin x = \sqrt{13}\cos(x + 56.3)^\circ$

b) $2\cos x - 3\sin x = 3$

Using the result from a) gives

$$\sqrt{13}\cos(x + 56.3)^\circ = 3$$

$$\cos(x + 56.3) = \frac{3}{\sqrt{13}}$$

$$cos^{-1}\left(\frac{3}{\sqrt{13}}\right) = 33.69^\circ$$

Using CAST, solutions to $\cos(x + 56.3) = \frac{3}{\sqrt{13}}$ should lie in quadrants 1 & 4

$$x + 56.3 = 33.69$$

$x = 33.69 - 56.3 = -22.61$ which is not in the range $0 \le x \le 360^\circ$

But since Cosine repeats values every 360° we have a solution given by

$$x = -22.61 + 360 = 337.39^\circ$$

$$x + 56.3 = 360 - 33.69$$

$$x = 270^\circ$$