Higher Mathematics

Optimization - **Solutions** - 2013-2019

Marks are indicated in brackets after each question number

2013 Paper 2 Question 7, (3) (7)

a) $L = 3x + 4y$ by inspection of the diagram

Area $= 24 = 2xy$

$$y = \frac{24}{2x} = \frac{12}{x}$$

Substituting $y = \dfrac{12}{x}$ into L gives

$$L = 3x + 4 \bullet \frac{12}{x} = 3x + \frac{48}{x}$$

b) i) $L(x) = 3x + \dfrac{48}{x}$

$$= 3x + 48x^{-1}$$

$$L'(x) = 3 - 48x^{-2}$$

$$= 3 - \frac{48}{x^2}$$

Minimum values occur where $L'(x) = 0$

$$3 - \frac{48}{x^2} = 0$$

$$3x^2 - 48 = 0$$

$$x^2 - 16 = 0$$

$$(x+4)(x-4) = 0$$

$$x = -4, x = 4$$

Since x is a length it cannot be negative, so $x = 4$

x	1	4	10
$L'(x)$	+	0	−
Shape	\	—	/

$x = 4$ gives a minimum value of $L(x)$

ii) Minimum length occurs where $x = 4$

$$L(4) = 3 \bullet 4 + \frac{48}{4} = 24m$$

Minimum cost $= 24 \bullet £8.25 = £198$

<u>2015 Paper 2 Question 8, (1) (1) (8)</u>

a) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

i) If the crocodile does not travel on land then $x = 20$

So, $T(20) = 5\sqrt{36 + 20^2} + 4(20 - 20)$

$= 5\sqrt{436} \cong 104$ tenths of a second, 10.4 seconds

ii) If the crocodile swims the shortest distance then $x = 0$

So, $T(0) = 5\sqrt{36 + 0^2} + 4(20 - 0)$

$= 110$ tenths of a second, 11 seconds

b) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

$$= 5(36 + x^2)^{\frac{1}{2}} + 80 - 4x$$

$$T'(x) = [\frac{5}{2}\left(36 + x^2\right)^{-\frac{1}{2}} \bullet 2x] - 4$$

$$= 5x(36 + x^2)^{-\frac{1}{2}} - 4$$

$$= \frac{5x}{(36 + x^2)^{\frac{1}{2}}} - 4$$

Minimum values occur where $T'(x) = 0$

Let $\frac{5x}{(36 + x^2)^{\frac{1}{2}}} - 4 = 0$

$$\frac{5x}{(36 + x^2)^{\frac{1}{2}}} = 4$$

$$4(36 + x^2)^{\frac{1}{2}} = 5x$$

$$(36 + x^2)^{\frac{1}{2}} = \frac{5x}{4}$$

Squaring both sides gives

$$36 + x^2 = \frac{25x^2}{16}$$

$$576 + 16x^2 = 25x^2$$

$$9x^2 - 576 = 0$$

$$x^2 - 64 = 0$$

$$x = 8, x = -8$$

Since x is a distance it cannot be negative so $x = 8$

We must prove that $x = 8$ does indeed give a minimum value of $T(x)$ using a nature table

x	1	8	10
$T'(x)$	$-$	0	$+$
Shape	\	—	/

So $x = 8$ does give a minimum value of $T(x)$

$T(8) = 5\sqrt{36 + 8^2} + 4(20 - 8)$

$= 98$

So, the minimum time possible is 98 tenths of a second, 9.8 seconds

<u>2016 Paper 2 Question 7, (3) (6)</u>

a) Length $= 9x + 8y$

Area $= 2y \bullet 3x = 108$

$6xy = 108$

$y = \dfrac{18}{x}$

Substituting $y = \dfrac{18}{x}$ into Length gives

Length $= 9x + 8 \bullet \dfrac{18}{x}$

$= 9x + \dfrac{144}{x}$

So, $L(x) = 9x + \dfrac{144}{x}$

b) Minimum values of $L(x)$ occur where $L'(x) = 0$

$L(x) = 9x + \dfrac{144}{x} = 9x + 144x^{-1}$

$L'(x) = 9 - 144x^{-2}$

$= 9 - \dfrac{144}{x^2}$

Let $L'(x) = 0$ to give

$9 - \dfrac{144}{x^2} = 0$

$9x^2 - 144 = 0$

$x^2 - 16 = 0$

$(x - 4)(x + 4) = 0$

$x = -4, x = 4$

Since x is a length it cannot be negative so $x = 4$

We have to show that $x = 4$ gives a minimum value of $L(x)$

x	1	4	10
$L'(x)$	$-$	0	$+$
Shape	\	—	/

So, $x = 4$ gives a minimum value of $L(x)$

<u>2018 Paper 2 Question 9, (6)</u>

$$P(x) = 2x + \frac{128}{x}$$

$$= 2x + 128x^{-1}$$

$$P'(x) = 2 - 128x^{-2} = 2 - \frac{128}{x^2}$$

For minimum values, $P'(x) = 0$

$$2 - \frac{128}{x^2} = 0$$

$$2x^2 - 128 = 0$$

$$x^2 = 64$$

$$x = -8, x = 8$$

Discard $x = -8$ since x cannot be negative

x	1	8	10
$P'(x)$	−	0	+
Shape	\	—	/

Substitute $x = 8$ to give

$$P(x) = (2\ x\ 8) + \frac{128}{8}$$

$$= 16 + 16$$

$$= 32$$

<u>2019 Paper 2 Question 11, (3) (6)</u>

a) Sides $= 4 \bullet 3x \bullet h$

$$= 12xh$$

Top $= (3x)^2 - x^2$

$$= 9x^2 - x^2$$

$$= 8x^2$$

Base $= 8x^2$

Inside of Tunnel $= 4xh$

Total $= 16x^2 + 12xh + 4xh$

$$= 16x^2 + 16xh \qquad (1)$$

Volume of cuboid including central tunnel $= l \bullet b \bullet h$

$$= (3x) \bullet (3x) \bullet h$$

$$= 9x^2h$$

Volume of tunnel $= l \bullet b \bullet h$

$= x^2 h$

Volume of cuboid without central tunnel $= 9x^2h - x^2h$

$= 8x^2h$

Since volume $= 2000$ we have

$2000 = 8x^2h$

$h = \dfrac{2000}{8x^2}$

Substituting into (1) gives Total Surface Area $= 16x^2 + 16x(\dfrac{2000}{8x^2})$

$= 16x^2 + \dfrac{4000}{x}$

So, $A(x) = 16x^2 + \dfrac{4000}{x}$

b) $A(x) = 16x^2 + \dfrac{4000}{x}$

$= 16x^2 + 4000x^{-1}$

For minimum areas $A'(x) = 0$ giving

$32x - 4000x^{-2} = 0$

$32x - \dfrac{4000}{x^2} = 0$

$32x^3 - 4000 = 0$

$x^3 = \dfrac{4000}{32}$

$x = \sqrt[3]{\dfrac{4000}{32}}$

$x = 5$

x	1	5	10
$A'(x)$	$-$	0	$+$
Shape	\	—	/

The nature table confirms that $x = 5$ gives a minimum value of $A(x)$

$$A(5) = 16 \bullet 5^2 + \frac{4000}{5}$$

$$= 400 + 800$$

$$= 1200\ cm^2$$