

Higher Mathematics

Integration - Solutions - 2013-2019

Marks are indicated in brackets after each question number

2013 Paper 1 Question 7, (2)

$$\begin{aligned}\int x(3x + 2) dx \\ &= \int (3x^2 + 2x) dx \\ &= x^3 + x^2 + c\end{aligned}$$

2013 Paper 1 Question 16, (2)

$$\begin{aligned}\int (1 - 6x)^{-\frac{1}{2}} dx \\ &= \frac{(1 - 6x)^{\frac{1}{2}}}{\frac{1}{2} \cdot (-6)} + c \\ &= -\frac{1}{3}(1 - 6x)^{\frac{1}{2}} + c\end{aligned}$$

2013 Paper 2 Question 4, (6)

Equation the line and curves equations gives

$$\begin{aligned}2x + 3 &= x^3 + 3x^2 + 2x + 3 \\ x^3 + 3x^2 &= 0 \\ x^2(x + 3) &= 0 \\ x = 0, x &= -3\end{aligned}$$

$$\text{When } x = -3, y = 2 \cdot (-3) + 3 = -3$$

$$\text{So } B = (-3, -3)$$

$$\begin{aligned}
\text{Area} &= \int_{-3}^0 (x^3 + 3x^2 + 2x + 3) - (2x + 3) dx \\
&= \int_{-3}^0 (x^3 + 3x^2) dx \\
&= \left[\frac{x^4}{4} + x^3 \right]_{-3}^0 \\
&= (0) - \left(\frac{81}{4} - 27 \right) = \frac{27}{4} \text{ units}^2
\end{aligned}$$

2013 Paper 2 Question 6, (5)

$$\int_0^a 5\sin 3x dx = \frac{10}{3}$$

$$\left[-\frac{5}{3}\cos 3x \right]_0^a = \frac{10}{3}$$

$$\left(-\frac{5}{3}\cos 3a \right) - \left(-\frac{5}{3}\cos 0 \right) = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a + \frac{5}{3} = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a = \frac{5}{3}$$

$$\cos 3a = -1$$

$$3a = \pi, a = \frac{\pi}{3}$$

There are no other solutions in $0 \leq a < \pi$

2014 Paper 1 Question 5, (2)

$$\begin{aligned} & \int (2x + 9)^5 dx \\ &= \frac{(2x + 9)^6}{6 \cdot 2} + c \\ &= \frac{(2x + 9)^6}{12} + c \end{aligned}$$

2014 Paper 2 Question 5, (5)

$$\int_4^t (3x + 4)^{-\frac{1}{2}} dx = 2$$

Using the reverse chain rule we have

$$\left[\frac{2}{3}(3x + 4)^{\frac{1}{2}} \right]_4^t = 2$$

$$\frac{2}{3}(3t + 4)^{\frac{1}{2}} - \frac{2}{3}(12 + 4)^{\frac{1}{2}} = 2$$

$$\frac{2}{3}(3t + 4)^{\frac{1}{2}} - \frac{8}{3} = 2$$

Simplifying gives

$$(3t + 4)^{\frac{1}{2}} = 7$$

$$3t + 4 = 49$$

$$t = 15$$

2014 Paper 2 Question 7, (5) (5)

a) $y = 2x$ (1)

$$y = 6x - x^2 \quad (2)$$

For intersection equate (1) & (2)

$$2x = 6x - x^2$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0, x = 4$$

$$\text{Area} = \int_0^4 [(6x - x^2) - 2x] dx$$

$$= \int_0^4 (4x - x^2) dx$$

$$= \left[2x^2 - \frac{1}{3}x^3 \right]_0^4$$

$$= \left(32 - \frac{64}{3} \right) - (0)$$

$$= \frac{96}{3} - \frac{64}{3} = \frac{28}{3} \text{ units}^2$$

$$\text{Area} = \frac{28}{3} \times 300\text{m}^2 = 2,800\text{m}^2$$

b) We must find the point of tangency of the line and the curve

Since the line is parallel to $y = 2x$ it has the same gradient, 2

So, let $\frac{d}{dx}(6x - x^2) = 2$ giving

$$6 - 2x = 2$$

$$x = 2$$

So, the point of tangency is at $x = 2$

When $x = 2$, $y = 6 \cdot 2 - 2^2 = 8$

So, the point of tangency is at (2, 8)

The equation of the 'road' line is given by $y = 2x + c$

To find c substitute (2, 8) which is a point on the line

$$8 = 2 \cdot 2 + c$$

$$c = 4$$

So $y = 2x + 4$

$$\text{Area} = \int_0^2 [(2x + 4) - (6x - x^2)] dx$$

$$= \int_0^2 (x^2 - 4x + 4) dx$$

$$= \left[\frac{1}{3}x^3 - 2x^2 + 4x \right]_0^2$$

$$= \left(\frac{8}{3} - 4 - 8 \right) - (0)$$

$$= \frac{20}{3} \text{ units}^2$$

$$\text{Area} = \frac{20}{3} \times 300 \text{m}^2 = 2,000 \text{m}^2$$

2015 Paper 1 Question 12, (4)

$$\begin{aligned}\int_0^{\frac{3\pi}{4}} (a \cos bx) dx &= \int_0^{\frac{\pi}{4}} (a \cos bx) dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (a \cos bx) dx \\ &= \frac{1}{2} - \left(2 \cdot \frac{1}{2}\right)\end{aligned}$$

Note that since the part of the graph between $\frac{\pi}{4}$ $\frac{3\pi}{4}$ is under the x axis it has a negative integral

$$= -\frac{1}{2}$$

2015 Paper 2 Question 4, (2) (7)

a) Equating $f(x)$ $g(x)$ gives

$$\frac{1}{4}x^2 - \frac{1}{2}x + 3 = \frac{1}{4}x^2 - \frac{3}{2}x + 5$$

Multiplying through by 4 gives

$$x^2 - 2x + 12 = x^2 - 6x + 20$$

$$4x = 8, x = 2$$

b) Area of half of the plaque is given by

$$\begin{aligned}&\int_0^2 f(x) - h(x) dx \\ &= \int_0^2 \left(\frac{1}{4}x^2 - \frac{1}{2}x + 3\right) - \left(\frac{3}{8}x^2 - \frac{9}{4}x + 3\right) dx \\ &= \int_0^2 \left(-\frac{1}{8}x^2 + \frac{7}{4}x\right) dx \\ &= \left[-\frac{1}{24}x^3 + \frac{7}{8}x^2\right]_0^2 \\ &= \left(-\frac{8}{24} + \frac{28}{8}\right) - (0)\end{aligned}$$

$$= \frac{19}{6} \text{ after simplifying}$$

$$\text{Multiply by 2 to get the total area} = \frac{19}{2} \text{ units}^2$$

2015 Paper 2 Question 7, (2) (2) (2)

$$\text{a) } \int (3\cos 2x + 1) dx = \frac{3}{2}\sin 2x + x + c$$

$$\begin{aligned} \text{b) } 3\cos 2x + 1 &= 3(\cos^2 x - \sin^2 x) + 1 \\ &= 3\cos^2 x - 3\sin^2 x + 1 \end{aligned}$$

Using $\cos^2 x + \sin^2 x = 1$ to give

$$\begin{aligned} &= 3\cos^2 x - 3\sin^2 x + \cos^2 x + \sin^2 x \\ &= 4\cos^2 x - 2\sin^2 x \end{aligned}$$

$$\text{c) } \int (\sin^2 x - 2\cos^2 x) dx$$

$$\begin{aligned} \sin^2 x - 2\cos^2 x &= -\frac{1}{2}(4\cos^2 x - 2\sin^2 x) \\ &= -\frac{1}{2}(3\cos 2x + 1) \\ &= -\frac{3}{2}\cos 2x - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{So, } \int (\sin^2 x - 2\cos^2 x) dx &= \int \left(-\frac{3}{2}\cos 2x - \frac{1}{2} \right) dx \\ &= -\frac{3}{4}\sin 2x - \frac{1}{2}x + c \end{aligned}$$

2016 Paper 1 Question 5, (2)

$$\begin{aligned} & \int 8\cos(4x + 1) dx \\ &= \frac{8\sin(4x + 1)}{4} + c \\ &= 2\sin(4x + 1) + c \end{aligned}$$

2016 Paper 2 Question 3, (2) (3) (1) (4)

a) i) Let $f(x) = 2x^3 - 9x^2 + 3x + 14$

Using synthetic division gives

-1	2	-9	3	14
		-2	11	-14
	2	-11	14	0

Since the remainder is zero $(x + 1)$ is a factor of $f(x)$

ii) $2x^3 - 9x^2 + 3x + 14 = 0$

$$(x + 1)(2x^2 - 11x + 14) = 0$$

$$(x + 1)(2x - 7)(x - 2) = 0$$

$$x = -1, x = 2, x = \frac{7}{2}$$

b) i) $y = 2x^3 - 9x^2 + 3x + 14$

$$y = (x + 1)(2x - 7)(x - 2)$$

By inspection of the roots $A = (-1, 0)$, $B = (2, 0)$

$$\begin{aligned}
\text{ii) Area} &= \int_{-1}^2 (2x^3 - 9x^2 + 3x + 14) dx \\
&= \left[\frac{2}{4}x^4 - \frac{9}{3}x^3 + \frac{3}{2}x^2 + 14x \right]_{-1}^2 \\
&= \left[\frac{1}{2}x^4 - 3x^3 + \frac{3}{2}x^2 + 14x \right]_{-1}^2 \\
&= (8 - 24 + 6 + 28) - \left(\frac{1}{2} + 3 + \frac{3}{2} - 14 \right) \\
&= 27
\end{aligned}$$

$$\text{Area} = 27 \text{ units}^2$$

2016 Paper 2 Question 9, (4)

$$f'(x) = \frac{2x + 1}{\sqrt{x}}$$

$$\begin{aligned}
f(x) &= \int f'(x) dx \\
&= \int \frac{2x + 1}{\sqrt{x}} dx \\
&= \int \frac{2x + 1}{x^{\frac{1}{2}}} dx \\
&= \int (2x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx \\
&= \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c
\end{aligned}$$

Since $f(9) = 40$ we have

$$40 = \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

$$40 = 36 + 6 + c$$

$$c = -2$$

$$\text{So, } f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2$$

2017 Paper 1 Question 10, (5) (4)

$$\begin{aligned} \text{a) Shaded Area} &= \int_0^2 (\text{top function} - \text{bottom function}) dx \\ &= \int_0^2 (x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1) dx \\ &= \int_0^2 (x^3 - 5x^2 + 6) dx \\ &= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 6x \right]_0^2 \\ &= \left(4 - \frac{40}{3} + 12 \right) - (0) \\ &= \frac{8}{3} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{b) Area} &= \int_0^2 (\text{top function} - \text{bottom function}) dx \\ &= \int_0^2 (1 - x) - (x^2 - 3x + 1) dx \\ &= \int_0^2 (2x - x^2) dx \\ &= \left[x^2 - \frac{1}{3}x^3 \right]_0^2 \\ &= \left(4 - \frac{8}{3} \right) - (0) \end{aligned}$$

$$= \frac{4}{3} \text{ units}^2$$

Since the total shaded area is $\frac{8}{3} \text{ units}^2$ $\frac{1}{2}$ of the shaded area is below the line $1 - x$

2017 Paper 1 Question 13, (4)

$$\int \frac{1}{(5 - 4x)^{\frac{1}{2}}} dx$$

$$= \int (5 - 4x)^{-\frac{1}{2}} dx$$

$$= \frac{(5 - 4x)^{\frac{1}{2}}}{\frac{1}{2} \cdot (-4)} + c$$

$$= -\frac{1}{8}(5 - 4x)^{\frac{1}{2}} + c$$