Higher Mathematics

Circle - **Solutions** - 2013-2019

Marks are indicated in brackets after each question

2013 Paper 1 Question 22, (2) (3) (3) (4)

$x^2 + y^2 + 2x + 4y - 27 = 0$

a) Centre $= (-1, -2)$, Radius $= \sqrt{1^2 + 2^2 + 27} = \sqrt{32}$

b) Let C be the centre of C_1

$m_{cp} = 1$

Therefore $m_{tan} = -1$ since perpendicular gradients

Using $y - b = m(x - a)$ with (3, 2) gives

$y - 2 = -1(x - 2)$

$y = -x + 5$

c) Radius of $c_2 = \dfrac{\sqrt{32}}{2} = \sqrt{8}$

Since the centre of $c_2 = (10, -1)$ we have

$(x - 10)^2 + (y + 1)^2 = (\sqrt{8})^2$

$(x - 10)^2 + (y + 1)^2 = 8$

Expanding brackets and collecting terms gives

$x^2 + y^2 - 20x + 2y + 93 = 0$

d) Substituting $y = -x + 5$ into c_2 gives

$x^2 + (5 - x)^2 - 20(5 - x) + 2y + 93 = 0$

Expanding and simplifying gives

$x^2 - 16x + 64 = 0$

Using the discriminant with $a = 1, b = -16, c = 64$ gives

$b^2 - 4ac = \left(-16\right)^2 - 4 \bullet 1 \bullet 64 = 0$

Since the discriminant equals 0 there is only one solution

Therefore, there is only one point at which the line $y = -x + 5$

Intersects with c_2. In other words, it is a tangent

2014 Paper 1 Question 2, (2)

$m_{CT} = \dfrac{2-(-1)}{1-3} = -\dfrac{3}{2}$

$m_{tan} = \dfrac{2}{3}$ since perpendicular

Using $y - b = m(x - a)$ with $(3, -1)$ gives

$y - (-1) = \dfrac{2}{3}(x - 3)$

$y = \dfrac{2}{3}x - 3$

2014 Paper 1 Question 23, (4) (3) (3)

a) $x^2 + y^2 + 2x - 4y - 15 = 0 \quad (1)$

Substitute $y = 3x - 5$ into (1)

$x^2 + (3x-5)^2 + 2(3x-5) - 4y - 15 = 0$

Simplifying gives

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, x = 3$

When $x = 1, y = -2$

When $x = 3, y = -4$

$P = (1, -2),\ Q = (3, -4)$

b) Centre of $C_1 = (1, -2)$

$$m_{QT} = \frac{4-2}{3-(-1)} = \frac{1}{2}$$

$$m_{PT} = \frac{2-(-2)}{-1-1} = -2$$

Since $m_{QT} \bullet m_{PT} = -1$ QT and PT are perpendicular

c) Since triangle PTQ is right-angled (from B) PQ is a diameter of C_2

Midpoint of PQ $= (2, 1) =$ centre of C_2

Distance from (2, 1) to Q $= \sqrt{(4-2)^2 + (3-1)^2} = \sqrt{10} =$ radius of C_2

Equation of C_2 is $(x-2)^2 + (y-1)^2 = (\sqrt{10})^2$

$$(x-2)^2 + (y-1)^2 = 10$$

<u>2014 Paper 2 Question 8, (5)</u>

$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$

Radius $= \sqrt{p^2 + (2p)^2 - (3p+2)}$

$= \sqrt{5p^2 - 3p - 2}$

Since the radius of a circle must be greater than zero we have

$\sqrt{5p^2 - 3p - 2} > 0$

$5p^2 - 3p - 2 > 0$

Let $5p^2 - 3p - 2 = 0$

$(5p+2)(p-1) = 0$

$p = -\frac{2}{5}, p = 1$

So, for $5p^2 - 3p - 2 > 0$ we have

$1 < p < -\frac{2}{5}$

2015 Paper 1 Question 11, (4) (6)

a) Circle centre $= (-8, -2)$, radius $= \sqrt{45}$

Let circle centre be C

$$m_{CT} = \frac{-2-(-5)}{-8-(-2)} = -\frac{1}{2}$$

$m_{tan} = 2$ since the tangent is perpendicular to CT

Using $y - b = m(x - a)$ with $(-2, -5)$ gives

$y - (-5) = 2(x - (-2))$

$y + 5 = 2x + 4$

$y = 2x - 1$

b) Consider where the line, $y = 2x - 1$, and the parabola meet

To find this point equate them to give

$2x - 1 = -2x^2 + px + 1 - p$

$2x^2 + 2x - px + p - 2 = 0$

Grouping terms gives

$2x^2 + (2-p)x + (p-2) = 0$

The solutions to this equation are the points where the line and parabola meet.

Since the line is a tangent to the parabola they meet at only one point.

$a = 2, b = (2-p), c = (p-2)$

Since there is one solution $b^2 - 4ac = 0$

$$b^2 - 4ac = (2-p)^2 - 4 \bullet 2 \bullet (p-2) = 0$$
$$4 - 4p + p^2 - 8p + 16 = 0$$
$$p^2 - 12p + 20 = 0$$
$$(p-10)(p-2) = 0$$
$$p = 2, p = 10$$

But since $p > 3$ the solution is $p > 10$

2015 Paper 1 Question 14, (2)

$x^2 + y^2 - 12x - 10y + k = 0$

Centre $= (6, 5)$ and Radius $= \sqrt{(-6)^2 + (-5)^2 - k}$

$= \sqrt{36 + 25 - k}$

$= \sqrt{51 - k}$

Mark the centre of the circle on an x-y axis. If the circle meets the axes at exactly three points it is soon seen that the radius $= 6$

So, $\sqrt{51 - k} = 6$

$51 - k = 36$

$k = 25$

2015 Paper 2 Question 5, (4) (4)

a) Centre of $C_1 = (-3, -5)$

Centre of $C_2 = (9, 11)$

Distance between $C_1\ C_2 = \sqrt{(11 - (-5))^2 + (9 - (-3))^2} = 20$

Radius of $C_1 = \sqrt{3^2 + 5^2 - 9} = 5$

Since the distance between the circle centres is 20 and the radius of C_1 is 5, the radius of $C_2 = 15$

b) The diameter of C_3 is equal to the diameter of $C_1 + C_2 = 40$. So, the radius of $C_3 = 20$

The centre of C_3 lies along the same line as the centres of $C_1\ C_2$ since they are collinear

By considering the radii of the circles we see that the centre of C_3 lies $\frac{3}{4}$ of the way along the line from the centre of C_1 to C_2 (use a sketch to confirm this)

The distance between the x co-ordinate of $C_1\ C_2$ is 12; $\frac{3}{4}$ of 12 = 9. So, starting at -3 and moving units gives 6.

The distance between the y co-ordinate of C_1 C_2 is 16; $\frac{3}{4}$ of 16 = 12. So, starting at -5 and moving 12 units gives 7.

So, the centre of $C_3 = (6, 7)$

Giving the equation of C_3 is $(x-6)^2 + (y-7)^2 = 20^2$

<u>2016 Paper 1 Question 4, (3)</u>

Diameter = distance between A & B

$$= \sqrt{(5-3)^2 + (1-(-7))^2}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Radius $= \frac{2\sqrt{17}}{2} = \sqrt{17}$

Circle centre = mid-point of AB

$$= (-3, 4)$$

So, the equation of the circle is $(x+3)^2 + (y-4)^2 = 17$

<u>2016 Paper 1 Question 8, (5)</u>

Substituting $y = 3x - 5$ into the circle equation gives

$$x^2 + (3x-5)^2 + 2x - 4(3x-5) - 4 = 0$$
$$x^2 + 9x^2 - 30x + 25 + 2x - 12x + 20 - 4 = 0$$
$$10x^2 - 40x + 40 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)(x-2) = 0$$

$x = 2$

Since there is only one point of intersection the line must be a tangent to the circle

When $x = 2$, $y = 1$

So, the point of tangency is $(2, 1)$

<u>2016 Paper 2 Question 4, (4) (3)</u>

a) For C_1 centre $= (-5, 6)$ radius $= 3$

For C_2 centre $= (3, 0)$ radius $= 5$

b) Distance between the centre of C_1 $C_2 = \sqrt{(6-0)^2 + (-5-3)^2}$

$$= \sqrt{36 + 64}$$

$$= 10$$

Since the sum of the radii $= 3 + 5 = 8 < 10$ the circles cannot intersect

<u>2017 Paper 1 Question 2, (4)</u>

$x^2 + y^2 - 8x - 6y - 15 = 0$

Centre $= (4, 3)$

Let $C = (4, 3)$

$$m_{CP} = \frac{3-1}{4-(-2)} = \frac{1}{3}$$

$m_{tan} = -3$ since perpendicular to CP

Using $y - b = m(x - a)$ with $(-2, 1)$ gives

$y - 1 = -3(x - (-2))$

$y = -3x - 5$

2017 Paper 2 Question 3, (5)

$(x-2)^2+(y-1)^2=25$

Substituting $y=3x$ into the circle equation gives

$(x-2)^2+(3x-1)^2=25$

Expanding brackets and simplifying gives

$x^2-2x-2=0$

$(x-2)(x+1)=0$

$x=-1, x=2$

When $x=-1, y=-3$ giving $(-1, -3)$

When $x=2, y=6$ giving $(2, 6)$

2017 Paper 2 Question 10, (3) (4)

a) $A=(-7, -2),\ B=(2, 1),\ C=(17, 6)$

$\vec{AB}=\begin{pmatrix}9\\3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}3\\1\end{pmatrix}$

$\vec{BC}=\begin{pmatrix}15\\5\end{pmatrix}=\frac{1}{5}\begin{pmatrix}3\\1\end{pmatrix}$

So, $\vec{AB}=\frac{3}{5}\vec{BC}$ showing that $\vec{AB}\ \ \vec{BC}$ are parallel and since B is a common point A, B, & C are collinear

b) $r_A=\sqrt{10}$

$r_B=2\sqrt{10}$

$r_C=r_A+r_B=3\sqrt{10}$

By inspection of the graph the radius of the circle with centre D is $6\sqrt{10}$

The point D divides the line AC in the ratio $5:3$

So, $\vec{AD}=\frac{5}{8}\vec{AC}=\frac{5}{8}\begin{pmatrix}24\\8\end{pmatrix}=\begin{pmatrix}15\\5\end{pmatrix}$

Then $D=(-7+15, -2+5)=(8, 3)$

So, the circle centre is (8, 3)

$$(x-8)^2+(y-3)^2=(6\sqrt{10})^2$$
$$(x-8)^2+(y-3)^2=360$$

<u>2018 Paper 1 Question 4, (4)</u>

$x^2+y^2-12x-6y-23=0$

Circle centre $=(6,3)$

Let $C=(6,3)$

$$m_{KC}=\frac{3-(-5)}{6-8}=\frac{8}{-2}=-4$$

$$m_{perp}=\frac{1}{4}$$

Using $y-b=m(x-a)$ with $(6,3)$ gives

$$y-3=\frac{1}{4}(x-6)$$

$$4y-12=x-6$$

$$4y=x+6$$

$$y=\frac{1}{4}x+\frac{6}{4}$$

<u>2018 Paper 2 Question 5, (3) (2) (2)</u>

a) Mid-point $PQ=(6,1)$

$$m_{PQ}=\frac{4\ +\ 2}{3\ -\ 9}=\frac{6}{-6}=-1$$

So, $m_{L1}=1$

$y - b = m(x - a)$

$y - 1 = x - 6$

$y = x - 5$

b) $3y + x = 25$

$3y = 25 - x$

$y = x - 5$

$3y = 3x - 15$

Equating gives

$3x - 15 = 25 - x$

$4x = 40$

$x = 10$

When $x = 10$, $y = 10 - 5 = 5$

So, $C = \left(10, 5\right)$

c) $C = \left(10, 5\right)$

$P = (3, 4)$

Distance $= \sqrt{(10 - 3)^2 + (5 - 4)^2}$

$= \sqrt{50}$

$(x-10)^2+(y-5)^2=50$

2018 Paper 2 Question 12, (1) (1) (2) (2) (1)

a) i) $(13,\ -4)$

ii) $x^2+y^2+14x-22y+c=0$

Substitute $(13,\ -4)$ to give

$13^2+\ (-4)^2+14(13)-22(-4)+c=0$

Simplify to give $c=-455$

b) i) C_1 radius $=10$

$C_2\ x^2+y^2+14x-22y-455=0$

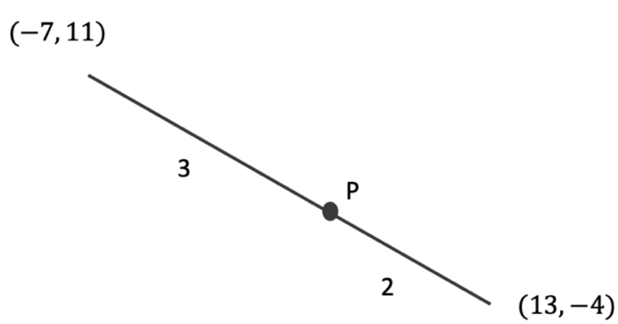
Radius $=\sqrt{7^2+11^2+455}$

$=25$

Ratio $=15:10$

ii) C_1 centre $=(13,\ -4)$

C_2 centre $=(-7,11)$



Using the stepping out technique:

For the x co-ordinate, $\frac{3}{5}\ x\ 20 = 12,\ -7 + 12 = 5$

For the y co-ordinate, $\frac{3}{5}\ x\ 15 = 9,\ 11 - 9 = 2$

$P = (5, 2)$

c) Radius $= 25 + 15 = 40$

Centre $= (5, 2)$

$(x - 5)^2 + (y - 2)^2 = 40^2$

$(x - 5)^2 + (y - 2)^2 = 1600$

<u>2019 Paper 1 Question 3, (2)</u>

Radius of $C_1 = \sqrt{(-3)^2 + 1^2 + 26}$

$= \sqrt{36}$

$= 6$

Radius of C_2 is the same as the radius of $C_1 = 6$

So, the equation of C_2 is $(x - 4)^2 + (y + 2)^2 = 6^2$

2019 Paper 1 Question 16, (2) (4)

a) $P = (4, k)$ $C = (1, -2)$

Using the distance formula gives, distance $= \sqrt{(4-1)^2 + (k+2)^2}$

$$= \sqrt{9 + k^2 + 4k + 4}$$

$$= \sqrt{k^2 + 4k + 13}$$

b) From the equation the circle has radius $= \sqrt{25} = 5$

If P lies outside the circle then the distance between C & P is greater than 5

So, $\sqrt{k^2 + 4k + 13} > 5$

$$k^2 + 4k + 13 > 25$$

$$k^2 + 4k - 12 > 0$$

$$(k+6)(k-2) > 0$$

By considering the roots and shape of the graph of $y = k^2 + 4k - 12$ we have

$k < -6$ and $k > 2$