2017 Higher Paper 1

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Paper 2: 1 2 3 4 5 6 7 8 9 10 11

Question 1

a) $f\left(g(x)\right) = f(2\cos x) = 5 \bullet 2 \cos x = 10\cos x$

$f\left(g(0)\right) = 10 \bullet \cos 0 = 10$

b) $g\left(f(x)\right) = g\left(5x\right) = 2\cos(5x)$

Question 2

$x^2 + y^2 - 8x - 6y - 15 = 0$

Centre $= (4, 3)$

Let $C = (4, 3)$

$$m_{CP} = \frac{3-1}{4-(-2)} = \frac{1}{3}$$

$m_{tan} = -3$ since perpendicular to CP

Using $y - b = m(x - a)$ with $(-2, 1)$ gives

$y - 1 = -3(x - (-2))$

$y = -3x - 5$

Question 3

$y = (4x - 1)^{12}$

$$\frac{dy}{dx} = 12(4x - 1)^{11} \bullet 4$$

$$= 48(4x - 1)^{11}$$

Question 4

$x^2 + 4x + (k - 5) = 0$

$a = 1, b = 4, c = k - 5$

For equal roots $b^2 - 4ac = 0$

$$4^2 - 4 \bullet 1 \bullet (k - 5) = 0$$

$$16 - 4k + 20 = 0$$

$$k = 9$$

Question 5

a) $\underline{u} \bullet \underline{v} = (5 \bullet 3) + (1 \bullet -8) + (-1 \bullet 6) = 1$

b) $\left|\underline{u}\right| = \sqrt{5^2 + 1^2 + (-1)^2} = \sqrt{27}$

$$\underline{u} \bullet \underline{w} = \left|\underline{u}\right|\left|\underline{w}\right|\cos(\frac{\pi}{3})$$

$$= \sqrt{27} \bullet \sqrt{3} \bullet \frac{1}{2}$$

$$= \sqrt{81} \bullet \frac{1}{2}$$

$$= \frac{9}{2}$$

Question 6

$h(x) = x^3 + 7$

Let $y = h(x)$

$y = x^3 + 7$

Swapping x and y gives

$x = y^3 + 7$

$y^3 = x - 7$

$y = \sqrt[3]{x - 7}$

So, $h^{-1}(x) = \sqrt[3]{x - 7}$

Question 7

Midpoint of AB $= (2, 7)$

Let M $= (2, 7)$

$$m_{CM} = \frac{11 - 7}{2 - 2} = \text{undefined}$$

So, the line is vertical and since it goes through (2, 7) has equation

$x = 2$

Question 8

$$d(t) = \frac{1}{2t} = \frac{t^{-1}}{2} = \frac{1}{2}t^{-1}$$

$$d'(t) = -\frac{1}{2}t^{-2} = -\frac{1}{2t^2}$$

$$d'(5) = -\frac{1}{2 \bullet 5^2} = -\frac{1}{50}$$

So, the rate of change of $d(t)$ when $t = 5$ is $-\dfrac{1}{50}$

Question 9

a) $u_{n+1} = mu_n + 6$

$u_2 = mu_1 + 6$

$13 = 28m + 6$

$7 = 28m$

$m = \dfrac{1}{4}$

b) i) Since $-1 < \dfrac{1}{4} < 1$ the sequence has a limit as $n \to \infty$

ii) Limit $= \dfrac{6}{1 - \frac{1}{4}} = \dfrac{6}{\frac{3}{4}} = 8$

Question 10

a) Shaded Area $= \int_0^2 \left(top\,function - bottom\,function\right)\,dx$

$$= \int_0^2 \left(x^3 - 4x^2 + 3x + 1\right) - \left(x^2 - 3x + 1\right)\,dx$$

$$= \int_0^2 (x^3 - 5x^2 + 6)\,dx$$

$$= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 6x]_0^2$$

$$= \left(4 - \frac{40}{3} + 12\right) - (0)$$

$$= \frac{8}{3}\,units^2$$

b) Area $= \int_0^2 \left(top\,function - bottom\,function\right)\,dx$

$$= \int_0^2 (1 - x) - \left(x^2 - 3x + 1\right)\,dx$$

$$= \int_0^2 \left(2x - x^2\right)\,dx$$

$$= [x^2 - \frac{1}{3}x^3]_0^2$$

$$= \left(4 - \frac{8}{3}\right) - (0)$$

$$= \frac{4}{3}\,units^2$$

Since the total shaded area is $\frac{8}{3}\,units^2$ $\frac{1}{2}$ of the shaded area is below the line $1 - x$

Question 11

$$3y - 2x = 4$$

$$y = \frac{2}{3}x + \frac{4}{3}$$

The gradient of this line is $\frac{2}{3}$

The gradient of the line joining A & B is also $\frac{2}{3}$ since parallel

$$m_{AB} = \frac{2 - a}{-7 - 5} = \frac{2}{3}$$

$$\frac{2 - a}{-12} = \frac{2}{3}$$

$$2 - a = -8$$

$$a = 10$$

Question 12

$$\log_a 36 - \log_a 4 = \frac{1}{2}$$

$$\log_a \left(\frac{36}{4}\right) = \frac{1}{2}$$

$$\log_a 9 = \frac{1}{2}$$

Rewrite as an exponential to give

$$a^{\frac{1}{2}} = 9$$

$$a = 9^2 = 81$$

Question 13

$$\int \frac{1}{(5 - 4x)^{\frac{1}{2}}}\, dx$$

$$= \int (5 - 4x)^{-\frac{1}{2}}\, dx$$

$$= \frac{(5 - 4x)^{\frac{1}{2}}}{\frac{1}{2} \bullet (-4)} + c$$

$$= -\frac{1}{8}(5-4x)^{\frac{1}{2}} + c$$

Question 14

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x-a)$

$$= k\sin x\cos a - k\cos x\sin a$$

$$= k\cos a\sin x - k\sin a\cos x$$

By inspection $\sqrt{3} = k\cos a$ and $1 = k\sin a$

$$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

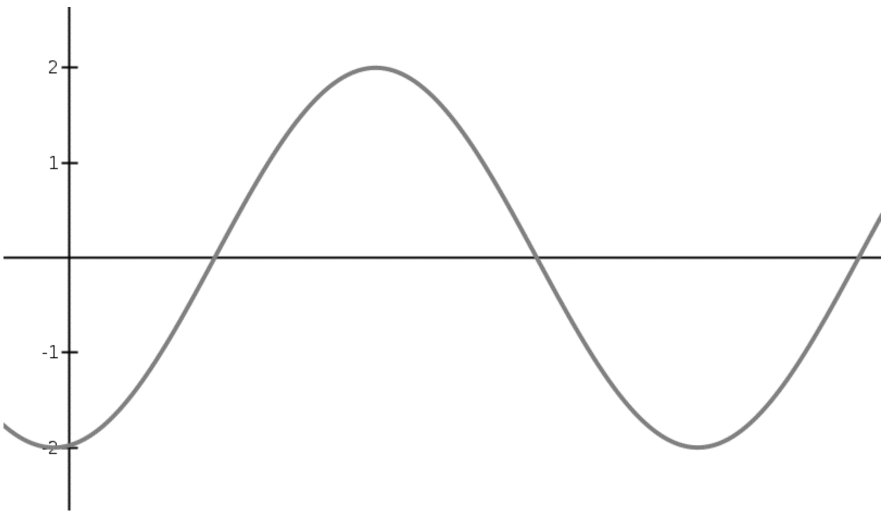
$$\frac{k\sin a}{k\cos a} = \frac{1}{\sqrt{3}} = \tan a$$

$$a = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

Confirm that this answer is in the correct quadrant using CAST

So, $\sqrt{3}\sin x - \cos x = 2\sin(x-30)°$

b) Since $\sqrt{3}\sin x - \cos x = 2\sin(x-30)°$ the graph of $y = \sqrt{3}\sin x - \cos x$ is the same as the graph of $y = 2\sin(x-30)$



Question 15

a) The graph of $h(x)$ has been shifted $+5$ units in the x direction and $+3$ units in the y direction. So, we have $a = -5, b = 3$

b) $\int_6^8 h(x)\,dx = \int_1^3 f(x)\,dx + (2.3)$ 2.3 is the area of the rectangle

$= 4 + 6 = 10$

c) $f'(1) = 6$ tells us that the gradient of the tangent line at $x = 1$ is 6.

By inspection of the symmetry of the graph of $h(x)$ we have $h'(8) = -f'(1) = -6$

Question 1

a) $m_{BC} = \dfrac{0-(-2)}{3-9} = -\dfrac{1}{3}$

$m_{perp} = 3$

Midpoint of BC $= (6, -1)$

Using $y - b = m(x-a)$ with $(6, -1)$ gives

$$y-(-1) = 3(x-1)$$

$$y = 3x - 19$$

b) $m_{AB} = \tan 45° = 1$

Using $y - b = m(x-a)$ with $(3, 0)$ gives

$$y - 0 = (x-3)$$

$$y = x - 3$$

c) Equating gives $3x - 19 = x - 3$

$$2x = 16$$

$$x = 8$$

When $x = 8, y = 5$

So, the point of intersection is $(8{,}50)$

Question 2

a) $f(x) = 2x^3 - 5x^2 + x + 2$

Using synthetic division gives

1	2	-5	1	2
		2	-3	-2
	2	-3	-2	0

Since the remainder is zero, $x - 1$ is a factor of $f(x)$

b) $f(x) = 2x^3 - 5x^2 + x + 2$

$= (x - 1)(2x^2 - 3x - 2)$

$= (x - 1)(2x + 1)(x - 2)$

For $f(x) = 0$ we have

$(x - 1)(2x + 1)(x - 2) = 0$

$x = -\frac{1}{2},\ x = 1, x = 2$

Question 3

$(x - 2)^2 + (y - 1)^2 = 25$

Substituting $y = 3x$ into the circle equation gives

$(x - 2)^2 + (3x - 1)^2 = 25$

Expanding brackets and simplifying gives

$x^2 - 2x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = -1, x = 2$

When $x = -1, y = -3$ giving $(-1,\ -3)$

When $x = 2, y = 6$ giving $\left(2, 6\right)$

Question 4

a) $3x^2 + 24x + 50 = 3(x^2 + 8x) + 50$

$$= 3\left[(x+4)^2 - 16\right] + 50$$

$$= 3(x+4)^2 - 48 + 50$$

$$= 3(x+4)^2 + 2$$

b) $f(x) = x^3 + 12x^2 + 50x - 11$

$f'(x) = 3x^2 + 24x + 50$

c) $f'(x) = 3x^2 + 24x + 50$

$= 3(x+4)^2 + 2$ (from a)

Since $(x+4)^2 \geq 0, 3(x+4)^2 + 2 > 0$

So, $f'(x) > 0$ meaning that $f(x)$ is strictly increasing for all x

Question 5

a) $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ} = \left(9\underline{i} + 5\underline{j} + 2\underline{k}\right) + \left(-12\underline{i} - 9\underline{j} + 3\underline{k}\right) = -3\underline{i} - 4\underline{j} + 5\underline{k}$

b) $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$

$$= \overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QR}$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + \frac{1}{3}(-\overrightarrow{RQ})$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + \frac{1}{3}(12\underline{i} + 9\underline{j} - 3\underline{k})$$

$$= -3\underline{i} - 4\underline{j} + 5\underline{k} + 4\underline{i} + 3\underline{j} - \underline{k}$$

$$= \underline{i} - \underline{j} + 4\underline{k}$$

c) $\overrightarrow{PQ} \bullet \overrightarrow{PS} = -3 + 4 + 20 = 21$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-3)^2 + (-4)^2 + 5^2} = \sqrt{50}$$

$$\left|\overrightarrow{PS}\right| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18}$$

Let θ be the angle between $\overrightarrow{PS}$ and $\overrightarrow{PQ} = QPS$

$$21 = \sqrt{50} \bullet \sqrt{18} \bullet \cos\theta$$

$$\frac{21}{\sqrt{50}\sqrt{18}} = \cos\theta$$

$$\theta = cos^{-1}(\frac{21}{\sqrt{50}\sqrt{18}})$$

$$\theta = 45.6°$$

Question 6

$5\sin x - 4 = 2\cos 2x$

$5\sin x - 4 = 2(1 - 2\sin^2 x)$ from double angle formula

$5\sin x - 4 = 2 - 4\ \sin^2 x$

Rearranging gives

$4\ \sin^2 x + 5\sin x - 6 = 0$

$(4\sin x - 3)(\sin x + 2) = 0$

Separating into two equations gives

$4\sin x - 3 = 0$ and $\sin x + 2 = 0$

$4\sin x - 3 = 0$

$\sin x = \frac{3}{4}$

$x = sin^{-1}\left(\frac{3}{4}\right) = 0.85$ radians

$x = \pi - 0.85 = 2.3$ radians

$\sin x + 2 = 0$

$\sin x = -2$

No solutions since $-1 \leq \sin x \leq 1$

Question 7

a) $y = 6x - 2\sqrt{x^3}$

$y = 6x - 2x^{\frac{3}{2}}$

$\frac{dy}{dx} = 6 - 3x^{\frac{1}{2}}$

For stationary points $\frac{dy}{dx} = 0$

$6 - 3x^{\frac{1}{2}} = 0$

$6 = 3x^{\frac{1}{2}}$

$2 = x^{\frac{1}{2}}$

Squaring both sides gives

$4 = x$

b) When $x = 4, y = 8$

When $x = 1, y = 4$

When $x = 9, y = 0$

So, the maximum value of y on $1 \leq x \leq 9$ is 8

And the minimum value is 0

Question 8

a) $u_{n+1} = ku_n - 20$

$u_1 = k \bullet u_0 - 20 = 5k - 20$

$u_2 = k \bullet u_1 - 20 = k(5k - 20) - 20$

$= 5k^2 - 20k - 20$

b) Let $u_2 < u_0$ to give

$5k^2 - 20k - 20 < 5$

$5k^2 - 20k - 25 < 0$

Let $5k^2 - 20k - 25 = 0$

$k^2 - 4k - 5 = 0$

$(k - 5)(k + 1) = 0$

$k = -1, k = 5$

So, $5k^2 - 20k - 25 < 0$ when $-1 \leq k \leq 5$

Question 9

$y = kx^n$

$\log_2 y = \log_2 kx^n$

$\log_2 y = \log_2 k + \log_2 x^n$

$\log_2 y = \log_2 k + \text{n}\log_2 x$

Reorder to give

$\log_2 y = \text{n}\log_2 x + \log_2 k$

n is the line gradient $= \dfrac{3 - 0}{0 - (-12)} = \dfrac{1}{4}$

$\log_2 k$ is the y intercept

$\log_2 k = 3$

$k = 2^3 = 8$

Question 10

a) $A = (-7,\ -2),\ B = (2, 1),\ C = (17, 6)$

$$\vec{AB} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

So, $\vec{AB} = \frac{3}{5}\vec{BC}$ showing that $\vec{AB}$ and $\vec{BC}$ are parallel and since B is a common point A, B, & C are collinear

b) $r_A = \sqrt{10}$

$r_B = 2\sqrt{10}$

$r_C = r_A + r_B = 3\sqrt{10}$

By inspection of the graph the radius of the circle with centre D is $6\sqrt{10}$

The point D divides the line AC in the ratio $5:3$

So, $\vec{AD} = \frac{5}{8}\vec{AC} = \frac{5}{8}\begin{pmatrix} 24 \\ 8 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$

Then $D = \left(-7 + 15,\ -2 + 5\right) = (8, 3)$

So, the circle centre is (8, 3)

$$(x-8)^2 + (y-3)^2 = (6\sqrt{10})^2$$
$$(x-8)^2 + (y-3)^2 = 360$$

Question 11

a) $$\frac{\sin 2x}{2\cos x} - \sin x\cos^2 x = \frac{2\sin x\cos x}{2\cos x} - \sin x\cos^2 x$$
$$= \sin x - \sin x\cos^2 x$$
$$= \sin x - \sin x\,(1 - \sin^2 x) \qquad \text{(since } \sin^2 x + \cos^2 x = 1\text{)}$$
$$= \sin x - \sin x + \sin^3 x$$
$$= \sin^3 x$$

b) $$\frac{d}{dx}\left(\frac{\sin 2x}{2\cos x} - \sin x\cos^2 x\right) = \frac{d}{dx}\left(\sin^3 x\right)$$
$$= \frac{d}{dx}(\sin x)^3$$
$$= 3(\sin x)^2 \bullet \cos x$$
$$= 3\cos x\,\sin^2 x$$