2016 Higher Paper 1

Question 1

$y + 4x = 7$

$y = -4x + 7$

Any line parallel to this has gradient of -4

Using $y - b = m(x - a)$ with $(-2, 3)$ gives

$y - 3 = -4(x - (-2)$

$y - 3 = -4(x + 2)$

$y = -4x - 5$

Question 2

$y = 12x^3 + 8\sqrt{x}$

$y = 12x^3 + 8x^{\frac{1}{2}}$

$\frac{dy}{dx} = 36x^2 + 4x^{-\frac{1}{2}}$

Question 3

a) $u_{n+1} = \frac{1}{3}u_n + 10$

$u_4 = \frac{1}{3}u_3 + 10$

$u_4 = \frac{1}{3} \bullet 6 + 10$

$u_4 = 12$

b) Since $-1 < \frac{1}{3} < 1$ the sequence has a limit as $n \to \infty$

c) Limit $= \dfrac{10}{1-\frac{1}{3}} = \dfrac{10}{\frac{2}{3}} = 15$

Question 4

Diameter = distance between A & B

$$= \sqrt{(5-3)^2 + (1-(-7))^2}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Radius $= \dfrac{2\sqrt{17}}{2} = \sqrt{17}$

Circle centre = mid-point of AB

$$= (-3, 4)$$

So, the equation of the circle is $(x+3)^2 + (y-4)^2 = 17$

Question 5

$$\int 8\cos(4x+1)\,dx$$

$$= \frac{8\sin(4x+1)}{4} + c$$

$$= 2\sin(4x+1) + c$$

Question 6

a) $f(x) = 3x + 5$

Let $y = 3x + 5$

Swap x and y to give

$x = 3y + 5$

$y = \dfrac{x-5}{3}$

So, $f^{-1}(x) = \dfrac{x-5}{3}$

b) $g^{-1}(7) = 2$

Question 7

a) $\overrightarrow{FH} = \overrightarrow{FG} + \overrightarrow{GH} = \underline{i} + 3\underline{j} - 4\underline{k}$

b) $\overrightarrow{FE} + \overrightarrow{FH} + \overrightarrow{HE} = \overrightarrow{FH} - \overrightarrow{EH} = -\underline{i} - 5\underline{k}$

Question 8

Substituting $y = 3x - 5$ into the circle equation gives

$x^2 + (3x-5)^2 + 2x - 4(3x-5) - 4 = 0$

$x^2 + 9x^2 - 30x + 25 + 2x - 12x + 20 - 4 = 0$

$10x^2 - 40x + 40 = 0$

$x^2 - 4x + 4 = 0$

$(x-2)(x-2) = 0$

$x = 2$

Since there is only one point of intersection the line must be a tangent to the circle

When $x = 2$, $y = 1$

So, the point of tangency is $(2, 1)$

Question 9

a) $f(x) = x^3 + 3x^2 - 24x$

$f'(x) = 3x^2 + 6x - 24$

For stationary points $f'(x) = 0$

Let $f'(x) = 0$

$3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4, x = 2$

b)

x	-10	-4	1	2	10
$f'(x)$	+	0	−	0	+
Shape	/	—	\	—	/

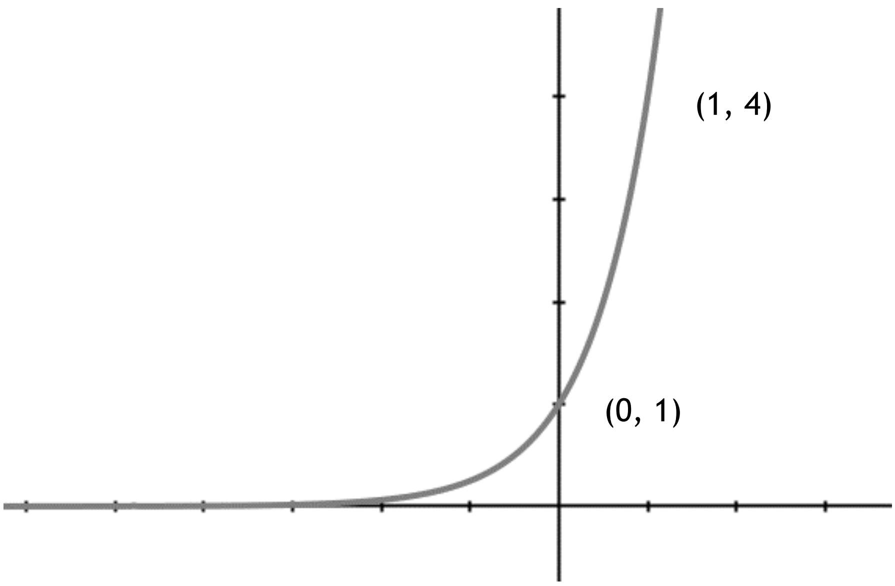
So, f is strictly increasing for $x < -4$ and for $x > 2$

So, $-4 > x > 2$

Question 10

The point (1, 0) on the graph of $f(x)$ is transformed to (0, 1) on the graph of $f^{-1}(x)$

The point (4, 1) on the graph of $f(x)$ is transformed to (1, 4) on the graph of $f^{-1}(x)$



Question 11

a) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

Since the ratio is $1:2$ there are 3 'parts' to the division

$$\frac{1}{3}\overrightarrow{AC} = \frac{1}{3}\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

So, $B = (1 + 1, 3 - 2, \; -2 + 2) = (2, 1, 0)$

b) $k\overrightarrow{AC} = \begin{pmatrix} 3k \\ -6k \\ 6k \end{pmatrix}$

$$\left| k\overrightarrow{AC} \right| = \sqrt{(3k)^2 + (-6k)^2 + (6k)^2}$$

$$= \sqrt{9k^2 + 36k^2 + 36k^2}$$

$$= \sqrt{81k^2}$$

$$= 9k$$

Since $\left| k\overrightarrow{AC} \right| = 1$ we have

$$9k = 1$$

$$k = 1$$

Question 12

a) $h(x) = f\big(g(x)\big) = f(3 - x) = 2(3 - x)^2 - 4(3 - x) + 5 = 2x^2 - 8x + 11$

b) $h(x) = 2x^2 - 8x + 11 = 2\left(x^2 - 4x\right) + 11$

$$= 2\left[(x - 2)^2 - 4\right] + 11$$

$$= 2(x - 2)^2 - 8 + 11$$

$$= 2(x - 2)^2 + 3$$

Question 13

$$\cos(q-p) = \cos p\cos q + \sin q\sin p$$

Use Pythagoras to work out the triangle hypotenuse

$$\cos(q-p) = \left(\frac{4}{5}\bullet\frac{4}{\sqrt{17}}\right) + (\frac{3}{5}\bullet\frac{1}{\sqrt{17}})$$

$$= \frac{16}{5\sqrt{17}} + \frac{3}{5\sqrt{17}}$$

$$= \frac{19}{5\sqrt{17}}$$

Multiply by $\frac{\sqrt{17}}{\sqrt{17}}$ to give

$$\cos(q-p) = \frac{19\sqrt{17}}{85}$$

Question 14

a) $\log_5 25 = 2$ since $5^2 = 25$

b) $\log_4 x + \log_4(x-6) = \log_5 25$

$$\log_4 x + \log_4(x-6) = 2$$

$$\log_4 x(x-6) = 2$$

Rewriting as an exponential gives

$$x(x-6) = 4^2$$

$$x^2 - 6x = 16$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0$$

$$x = -2, x = 8$$

Question 15

a) $f(x) = k(x - a)(x - b)^2$

By inspection of the roots we have

$f(x) = k(x - 4)(x + 5)^2$

Substituting the point (1, 9) gives

$9 = k(1 - 4)(1 + 5)^2$

Rearranging and solving for k gives

$$k = -\frac{1}{12}$$

b) $g(x) = f(x) - d$

Since d is positive the graph of $g(x)$ is the graph of $f(x)$ moved down by $'d'$ units

By considering the graph of $f(x)$ it must be moved down by at least 9 units to give only one root - in other words to move the point (1, 9) below the x axis.

So, $d > 9$

Question 1

a) i) $M = (2, 4)$

ii) $m_{PM} = \dfrac{-4-4}{0-2} = 4$

Using $y - b = m(x - a)$ with $(2{,}4)$ gives

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

b) $m_{PR} = \dfrac{6-(-4)}{10-0} = 1$

So, $m_{perp} = -1$

Using $y - b = m(x - a)$ with $(2{,}4)$ gives

$$y - 4 = -1(x - 2)$$

$$y = -x + 6$$

c) Midpoint of PR $= (5, 1)$

Test whether $(5{,}1)$ lies on L

Substituting $x = 5$ into L gives

$y = -5 + 6 = 1$

So, $(5{,}1)$ does lie on the line L

Question 2

$x^2 - 2x + 3 - p = 0$

$a = 1, b = -2, c = 3 - p$

For no real roots $b^2 - 4ac < 0$

$$b^2 - 4ac < 0$$

$$(-2)^2 - 4 \bullet 1 \bullet (3 - p) < 0$$

$$4 - 12 + 4p < 0$$

$$4p < 8, p < 2$$

Question 3

a) i) Let $f(x) = 2x^3 - 9x^2 + 3x + 14$

Using synthetic division gives

-1	2	-9	3	14
		-2	11	-14
	2	-11	14	0

Since the remainder is zero $(x+1)$ is a factor of $f(x)$

ii) $2x^3 - 9x^2 + 3x + 14 = 0$

$$(x+1)\left(2x^2 - 11x + 14\right) = 0$$

$$(x+1)(2x-7)(x-2) = 0$$

$$x = -1, x = 2, x = \frac{7}{2}$$

b) i) $y = 2x^3 - 9x^2 + 3x + 14$

$$y = (x+1)(2x-7)(x-2)$$

By inspection of the roots $A = (-1, 0),\ B = (2, 0)$

ii) Area $= \displaystyle\int_{-1}^{2} \left(2x^3 - 9x^2 + 3x + 14\right) dx$

$$= [\frac{2}{4}x^4 - \frac{9}{3}x^3 + \frac{3}{2}x^2 + 14x]_{-1}^{2}$$

$$= [\frac{1}{2}x^4 - 3x^3 + \frac{3}{2}x^2 + 14x]_{-1}^{2}$$

$$= \left(8 - 24 + 6 + 28\right) - (\frac{1}{2} + 3 + \frac{3}{2} - 14)$$

$$= 27$$

Area $= 27\ units^2$

Question 4

a) For C_1 centre $= (-5, 6)$ radius $= 3$

For C_2 centre $= (3, 0)$ radius $= 5$

b) Distance between the centre of C_1 $C_2 = \sqrt{(6-0)^2 + (-5-3)^2}$

$$= \sqrt{36+64}$$

$$= 10$$

Since the sum of the radii $= 3 + 5 = 8 < 10$ the circles cannot intersect

Question 5

a) $\vec{AB} = \begin{pmatrix} -8 \\ -16 \\ -2 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -2 \\ -8 \\ -16 \end{pmatrix}$

b) $\vec{AB} \bullet \vec{AC} = (-8 \bullet -2) + \left(-16 \bullet -8\right) + (-2 \bullet -16)$

$$= 112$$

$$\left|\vec{AB}\right| = \sqrt{(-8)^2 + (-16)^2 + (-2)^2} = 18$$

$$\left|\vec{AC}\right| = \sqrt{(-2)^2 + (-8)^2 + (-16)^2} = 18$$

Let θ be the angle BAC

$$\vec{AB} \bullet \vec{AC} = \left|\vec{AB}\right| \left|\vec{AC}\right| \cos\theta$$

$$112 = 18 \bullet 18\cos\theta$$

$$112 = 324\cos\theta$$

$$\frac{112}{324} = \cos\theta$$

$$\theta = cos^{-1}\left(\frac{112}{324}\right) = 69.8^\circ$$

Question 6

a) $B(t) = 200e^{0.107t}$

At the start $t = 0$ giving $B(0) = 200 \bullet e^0 = 200$

b) When the number doubles there are 400 bacteria

Let $B(t) = 400$

$$400 = 200e^{0.107t}$$

$$2 = e^{0.107t}$$

$$\log_e 2 = \log_e e^{0.107t}$$

$$\log_e 2 = 0.107t$$

$$t = \frac{\log_e 2}{0.107} = 6.48 \text{ hours}$$

Question 7

a) Length $= 9x + 8y$

Area $= 2y \bullet 3x = 108$

$$6xy = 108$$

$$y = \frac{18}{x}$$

Substituting $y = \frac{18}{x}$ into Length gives

Length $= 9x + 8 \bullet \frac{18}{x}$

$$= 9x + \frac{144}{x}$$

So, $L(x) = 9x + \frac{144}{x}$

b) Minimum values of $L(x)$ occur where $L'(x) = 0$

$$L(x) = 9x + \frac{144}{x} = 9x + 144x^{-1}$$

$$L'(x) = 9 - 144x^{-2}$$

$$= 9 - \frac{144}{x^2}$$

Let $L'(x) = 0$ to give

$$9 - \frac{144}{x^2} = 0$$

$$9x^2 - 144 = 0$$

$$x^2 - 16 = 0$$

$$(x-4)(x+4) = 0$$

$$x = -4, x = 4$$

Since x is a length it cannot be negative so $x = 4$

We have to show that $x = 4$ gives a minimum value of $L(x)$

x	1	4	10
$L'(x)$	$-$	0	$+$
$Shape$	↘	—	↗

So, $x = 4$ gives a minimum value of $L(x)$

Question 8

a) Let $5\cos x - 2\sin x = k\cos(x + a)$

$$= k\cos x\cos a - k\sin a\sin x$$

$$= k\cos a\cos x - k\sin a\sin x$$

By inspection $k\cos a = 5$ and $k\sin a = 2$

$k = \sqrt{5^2 + 2^2} = \sqrt{29}$

$$\frac{k\sin a}{k\cos a} = \frac{2}{5} = \tan a$$

$a = tan^{-1}\left(\frac{2}{5}\right) = 0.38$ radians

So, $5\cos x - 2\sin x = \sqrt{29}\cos(x + 0.38)$

b) Equating the line and curve gives

$10 + 5\cos x - 2\sin x = 12$

$5\cos x - 2\sin x = 2$

Using the result from part a) we have

$\sqrt{29}\cos(x + 0.38) = 2$

$$\cos(x + 0.38) = \frac{2}{\sqrt{29}}$$

First solution

$$x + 0.38 = cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$$

$x + 0.38 = 1.19$

$x = 0.81$ radians

Second solution

$x + 0.38 = 2\pi - 0.81$

$x = 4.71$ radians

Question 9

$$f'(x) = \frac{2x + 1}{\sqrt{x}}$$

$$f(x) = \int f'(x)\,dx$$

$$= \int \frac{2x+1}{\sqrt{x}}\, dx$$

$$= \int \frac{2x+1}{x^{\frac{1}{2}}}\, dx$$

$$= \int (2x^{\frac{1}{2}} + x^{-\frac{1}{2}})\, dx$$

$$= \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

Since $f(9) = 40$ we have

$$40 = \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

$$40 = 36 + 6 + c$$

$$c = -2$$

So, $f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2$

Question 10

a) $y = (x^2+7)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(x^2+7)^{-\frac{1}{2}} \bullet 2x$$

$$= x(x^2+7)^{-\frac{1}{2}}$$

$$= \frac{x}{(x^2+7)^{\frac{1}{2}}}$$

$$= \frac{x}{\sqrt{x^2+7}}$$

b) $\int \frac{4x}{\sqrt{x^2+7}}\, dx$

$$= 4\int \frac{x}{\sqrt{x^2+7}}\, dx$$

$= 4(x^2 + 7)^{\frac{1}{2}}$ using the result from a)

Question 11

a) $\sin 2x \tan x = 2\sin x \cos x \tan x$

$$= 2\sin x \cos x \frac{\sin x}{\cos x}$$

$$= 2\sin^2 x$$

$$= 1 - 2\cos 2x$$ (from double angle formula)

b) $f(x) = \sin 2x \tan x$

$$= 1 - 2\cos 2x$$

$$f'(x) = 4\sin 2x$$