2015 Higher Paper 1

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Question 1

Since $\underline{u}$ & $\underline{v}$ are perpendicular $\underline{u} \bullet \underline{v} = 0$

$$\underline{u} \bullet \underline{v} = \left(8 \bullet (-3)\right) + (2 \bullet t) + \left((-1) \bullet \left(-6\right)\right) = 0$$

$$-24 + 2t + 6 = 0$$

$$2t = 18$$

$$t = 9$$

Question 2

$$y = 2x^3 + 3$$

$$\frac{dy}{dx} = 6x^2$$

When $x = -2$, $\frac{dy}{dx} = 6 \bullet (-2)^2 = 24$

So, gradient of tangent at $x = -2$ is 24

When $x = -2$, $y = 2 \bullet (-2)^3 + 3 = -13$

So, $(-2, -13)$ lies on the curve

Using $y - b = m(x - a)$ we have

$$y - (-13) = -2(x - (-13))$$

$$y + 13 = -2x - 26$$

$$y = -2x - 39$$

Question 3

Let $f(x) = x^3 - 3x^2 - 10x + 24$

Using synthetic division gives

-3	1	-3	-10	24
		-3	18	-24
	1	-6	8	0

Since the remainder is zero, $(x + 3)$ is a factor of $f(x)$

$$f(x) = (x + 3)(x^2 - 6x + 8)$$
$$= (x + 3)(x - 4)(x - 2)$$

Question 4

The amplitude is 3 so $p = 3$

The period is $180°$ so $q = 2$

The graph has been shifted up by ‘1’ unit after p has been applied, so $r = 1$

Question 5

a) $g(x) = 6 - 2x$

$y = 6 - 2x$

Interchange x and y to give

$x = 6 - 2y$

Solve for y

$$y = 3 - \frac{1}{2}x$$

So, $g^{-1}(x) = 3 - \frac{1}{2}x$

b) $g\left(g^{-1}(x)\right) = x$ since g and g^{-1} are inverses of each other

Question 6

$$\log_6 12 + \frac{1}{3}\log_6 27 = \log_6 12 + \log_6 27^{\frac{1}{3}}$$
$$= \log_6 12 + \log_6 3$$
$$= \log_6 36$$
$$= 2$$

Question 7

$$f(x) = \sqrt{x}(3x - \frac{2}{x\sqrt{x}})$$
$$= x^{\frac{1}{2}}(3x - \frac{2}{x \bullet x^{\frac{1}{2}}})$$
$$= x^{\frac{1}{2}}(3x - \frac{2}{x^{\frac{3}{2}}})$$
$$= x^{\frac{1}{2}}(3x - 2x^{-\frac{3}{2}})$$
$$= 3x^{\frac{3}{2}} - 2x^{-1}$$

$$f'(x) = \frac{9}{2}x^{\frac{1}{2}} + 2x^{-2} = \frac{9}{2}x^{\frac{1}{2}} + \frac{2}{x^2}$$

$$f'(4) = \left(\frac{9}{2} \bullet 2\right) + \frac{2}{16} = 9 + \frac{1}{8} = \frac{73}{8}$$

Question 8

Area $= x(x-2) < 15$

$$x^2 - 2x - 15 < 0$$

$$(x-5)(x+3) < 0$$

Consider $\left(x-5\right)(x+3) = 0$

$$x = -3, x = 5$$

Since the graph of $x^2 - 2x - 15$ is U-shaped with roots at $x = -3, x = 5$ we have $(x-5)(x+3) < 0$ for $-3 < x < 5$

Question 9

$$y + \sqrt{3}x = 0$$

$$y = -\sqrt{3}x$$

So, AB has a gradient of $-\sqrt{3}$ since parallel

$$\tan 150° = -\tan 30° = -\frac{\sqrt{3}}{3}$$

So, BC has a gradient of $-\frac{\sqrt{3}}{3}$

Thus A,B, C cannot be collinear

Question 10

a) Using right-triangle with sides, 3,4, & 5 we have

$$\cos 2x = \frac{4}{5}$$

b) $\cos 2x = 2\cos^2 x - 1$ using the double angle formula

$$\frac{4}{5} = 2(\cos x)^2 - 1$$

$$\frac{9}{5} = 2(\cos x)^2$$

$$\frac{9}{10} = (\cos x)^2$$

$$\cos x = \sqrt{\frac{9}{10}}$$

Question 11

a) Circle centre $= (-8, -2)$, radius $= \sqrt{45}$

Let circle centre be C

$$m_{CT} = \frac{-2 - (-5)}{-8 - (-2)} = -\frac{1}{2}$$

$m_{tan} = 2$ since the tangent is perpendicular to CT

Using $y - b = m(x - a)$ with $(-2, -5)$ gives

$$y - (-5) = 2(x - (-2))$$

$$y + 5 = 2x + 4$$

$$y = 2x - 1$$

b) Consider where the line, $y = 2x - 1$, and the parabola meet

To find this point equate them to give

$$2x - 1 = -2x^2 + px + 1 - p$$

$$2x^2 + 2x - px + p - 2 = 0$$

Grouping terms gives

$$2x^2 + (2 - p)x + (p - 2) = 0$$

The solutions to this equation are the points where the line and parabola meet.

Since the line is a tangent to the parabola they meet at only one point.

$$a = 2, b = (2 - p),\ c = (p - 2)$$

Since there is one solution $b^2 - 4ac = 0$

$$b^2 - 4ac = (2 - p)^2 - 4 \bullet 2 \bullet (p - 2) = 0$$

$$4 - 4p + p^2 - 8p + 16 = 0$$

$$p^2 - 12p + 20 = 0$$

$$(p - 10)(p - 2) = 0$$

$$p = 2, p = 10$$

But since $p > 3$ the solution is $p > 10$

Question 12

$$\int_0^{\frac{3\pi}{4}} (a\cos bx)\ dx = \int_0^{\frac{\pi}{4}} (a\cos bx)\ dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (a\cos bx)\ dx$$

$$= \frac{1}{2} - (2 \bullet \frac{1}{2})$$

Note that since the part of the graph between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is under the x axis it has a negative integral

$$= -\frac{1}{2}$$

Question 13

a) $f(x) = 2^x + 3$

Substitute $x = 1$ to give

$f(1) = 2^1 + 3 = 5$

So, $b = 5$

b) i) The graph would be mirrored in the $45°$ line $y = x$ with all of the x and y co-ordinates swapped

ii) The image of $P = (5, 1)$

The co-ordinate of $Q = (0, 4)$ so the image of $Q = (4, 0)$

c) $y = 4 - f(x + 1)$

$= -f(x + 1) + 4$

The x co-ordinate of the image of R is 2

The y co-ordinate of the image of R is $-11 + 4 = -7$

So, the co-ordinates of the image of R $= (2, -7)$

Question 14

$x^2 + y^2 - 12x - 10y + k = 0$

Centre $= (6, 5)$ and Radius $= \sqrt{(-6)^2 + (-5)^2 - k}$

$= \sqrt{36 + 25 - k}$

$= \sqrt{51 - k}$

Mark the centre of the circle on an x-y axis. If the circle meets the axes at exactly three points it is soon seen that the radius $= 6$

So, $\sqrt{51 - k} = 6$

$51 - k = 36$

$k = 25$

Question 15

$$\frac{dT}{dt} = \frac{1}{25}t - k$$

$$T(t) = \int \frac{dT}{dt}\,dt$$

$$= \int (\frac{1}{25}t - k)\,dt$$

$$= \frac{1}{50}t^2 - kt + c$$

Now we must find k and c using the information given in the question

At $t = 0, T = 100$ (note that 'initially' means $t = 0$)

So, $100 = \left(\frac{1}{50} \bullet 0\right) - \left(k \bullet 0\right) + c$

$c = 100$

At $t = 10, T = 82$

So, $82 = \left(\frac{1}{50} \bullet 10^2\right) - 10k + 100$

$$82 = 2 - 10k + 100$$

$$10k = 20$$

$$k = 2$$

So, $T(t) = = \frac{1}{50}t^2 - 2t + 100$

2015 Higher Paper 2

Question 1

a) $m_{AB} = \dfrac{7 - (-5)}{-5 - (-1)} = -3$

$m_{alt} = \dfrac{1}{3}$ since perpendicular to AB

Using $y - b = m(x - a)$ with $(13, 3)$ gives

$$y - 3 = \frac{1}{3}(x - 13)$$

Rearranging gives

$$y = \frac{1}{3}x + \frac{26}{3}$$

b) Mid-point of AC $= (4, 5)$ by inspection

$$m_{med} = \frac{5 - (-5)}{4 - (-1)} = 2$$

Using $y - b = m(x - a)$ with $(4, 5)$ we have

$$y - 5 = 2(x - 4)$$

$$y = 2x - 3$$

c) Equating the two lines gives

$$\frac{1}{3}x + \frac{26}{3} = 2x - 3$$

$$x + 26 = 6x - 9$$

$$5x = 35$$

$$x = 7$$

When $x = 7, y = 11$

So, the point of intersection is $(7, 11)$

Question 2

a) $f\left(g(x)\right) = f((1+x)(3-x)+2)$

$$= 10 + (1+x)(3-x) + 2$$

Simplifying gives

$$f\left(g(x)\right) = 15 + 2x - x^2$$

b) $f\left(g(x)\right) = -x^2 + 2x + 15$

$$= -\left(x^2 - 2x\right) + 15$$

Completing the square of the inside of the bracket

$$= -\left[(x-1)^2 - 1\right] + 15$$

$$= -(x-1)^2 + 16$$

c) $h(x) = \dfrac{1}{f(g(x))} = \dfrac{1}{-(x-1)^2 + 16}$

Restrictions on the domain of $h(x)$ occur where $-(x-1)^2 + 16 = 0$

Let $-(x-1)^2 + 16 = 0$

$$(x-1)^2 = 16$$

$$x - 1 = 4 \qquad x - 1 = -4$$

$$x = 5 \qquad x = -3$$

So, $x = -3, x = 5$ cannot be in the domain of $h(x)$

Question 3

a) $t_2 = \dfrac{3}{4}t_1 + 13$

$$= \frac{3}{4} \bullet 13 + 13$$

$$= \frac{39}{4} + \frac{52}{4} = \frac{91}{4}$$

b) $f_{n+1} = \frac{1}{3} f_n + 32$

Limit $= \frac{32}{1 - \frac{1}{3}} = \frac{32}{\frac{2}{3}} = 48$

Since $48 < 50$ the frog does not escape from the well

$t_{n+1} = \frac{3}{4} t_n + 13$

Limit $= \frac{13}{1 - \frac{3}{4}} = \frac{13}{\frac{1}{4}} = 52$

Since $52 > 50$ the toad does escape from the well

Question 4

a) Equating $f(x) \quad g(x)$ gives

$$\frac{1}{4}x^2 - \frac{1}{2}x + 3 = \frac{1}{4}x^2 - \frac{3}{2}x + 5$$

Multiplying through by 4 gives

$$x^2 - 2x + 12 = x^2 - 6x + 20$$

$$4x = 8$$

$$x = 2$$

b) Area of half of the plaque is given by

$$\int_0^2 f(x) - h(x)\, dx$$

$$= \int_0^2 (\frac{1}{4}x^2 - \frac{1}{2}x + 3) - (\frac{3}{8}x^2 - \frac{9}{4}x + 3)\, dx$$

$$= \int_0^2 \left(-\frac{1}{8}x^2 + \frac{7}{4}x\right) dx$$

$$= \left[-\frac{1}{24}x^3 + \frac{7}{8}x^2\right]_0^2$$

$$= \left(-\frac{8}{24} + \frac{28}{8}\right) - (0)$$

$= \dfrac{19}{6}$ after simplifying

Multiply by 2 to get the total area $= \dfrac{19}{2} units^2$

Question 5

a) Centre of $C_1 = (-3, -5)$

Centre of $C_2 = (9, 11)$

Distance between C_1 and $C_2 = \sqrt{(11-(-5))^2 + (9-(-3))^2} = 20$

Radius of $C_1 = \sqrt{3^2 + 5^2 - 9} = 5$

Since the distance between the circle centres is 20 and the radius of C_1 is 5, then the radius of $C_2 = 15$

b) The diameter of C_3 is equal to the diameter of $C_1 + C_2 = 40$. So, the radius of $C_3 = 20$

The centre of C_3 lies along the same line as the centres of C_1 and C_2 since they are collinear

By considering the radii of the circles we see that the centre of C_3 lies $\dfrac{3}{4}$ of the way along the line from the centre of C_1 to C_2 (use a sketch to confirm this)

The distance between the x co-ordinate of C_1 and C_2 is 12; $\dfrac{3}{4}$ of 12 $= 9$. So, starting at -3 and moving 9 units gives 6

The distance between the y co-ordinate of C_1 C_2 is 16; $\frac{3}{4}$ of 16 $= 12$. So, starting at -5 and moving 12 units gives 7

So, the centre of $C_3 = (6, 7)$

Giving the equation of C_3 is $(x-6)^2 + (y-7)^2 = 20^2$

Question 6

a) $\underline{p} \bullet \left(\underline{q} + \underline{r}\right) = \underline{p} \bullet \underline{q} + \underline{p} \bullet \underline{r}$

$$= \left(\left|\underline{p}\right|\left|\underline{q}\right|\cos 60°\right) + (\left|\underline{p}\right|\left|\underline{r}\right|\cos 90°)$$

$$= \left(3 \bullet 3 \bullet \frac{1}{2}\right) + 0 = \frac{9}{2}$$

b) $\overrightarrow{EC} = -\underline{q} + \underline{p} + \underline{r}$

c) $\overrightarrow{AE} \bullet \overrightarrow{EC} = \underline{q} \bullet (-\underline{q} + \underline{p} + \underline{r})$

$$= -\underline{q} \bullet \underline{q} + \underline{q} \bullet \underline{p} + \underline{q} \bullet \underline{r}$$

$$= -\left|\underline{q}\right|\left|\underline{q}\right|\cos 0 + \frac{9}{2} + \left|\underline{q}\right|\left|\underline{r}\right|\cos 30°$$

(to get the $30°$ consider the diagram carefully!)

$$= -9 + \frac{9}{2} + \frac{3\sqrt{3}}{2}\left|\underline{r}\right|$$

But since $\overrightarrow{AE} \bullet \overrightarrow{EC} = 9\sqrt{3} - \frac{9}{2}$ we have

$$9\sqrt{3} - \frac{9}{2} = -9 + \frac{9}{2} + \frac{3\sqrt{3}}{2}\left|\underline{r}\right|$$

Multiplying through by 2 gives

$$18\sqrt{3} - 9 = -18 + 9 + 3\sqrt{3}\left|\underline{r}\right|$$

$$18\sqrt{3} = 3\sqrt{3}\left|\underline{r}\right|$$

$$\left|\underline{r}\right| = 6$$

Question 7

a) $\int (3\cos 2x + 1)\, dx = \frac{3}{2}\sin 2x + x + c$

b) $3\cos 2x + 1 = 3\left(\cos^2 x - \sin^2 x\right) + 1$

$$= 3\cos^2 x - 3\sin^2 x + 1$$

Use $\cos^2 x + \sin^2 x = 1$ to give

$$= 3\cos^2 x - 3\sin^2 x + \cos^2 x + \sin^2 x$$

$$= 4\cos^2 x - 2\sin^2 x$$

c) $\int (\sin^2 x - 2\cos^2 x)\, dx$

$$\sin^2 x - 2\cos^2 x = -\frac{1}{2}(4\cos^2 x - 2\sin^2 x)$$

$$= -\frac{1}{2}(3\cos 2x + 1)$$

$$= -\frac{3}{2}\cos 2x - \frac{1}{2}$$

So, $\int (\sin^2 x - 2\cos^2 x)\, dx = \int \left(-\frac{3}{2}\cos 2x - \frac{1}{2}\right) dx$

$$= -\frac{3}{4}\sin 2x - \frac{1}{2}x + c$$

Question 8

a) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

i) If the crocodile does not travel on land then $x = 20$

So, $T(20) = 5\sqrt{36 + 20^2} + 4(20 - 20)$

$= 5\sqrt{436} \cong 104$ tenths of a second, 10.4 seconds

ii) If the crocodile swims the shortest distance then $x = 0$

So, $T(0) = 5\sqrt{36 + 0^2} + 4(20 - 0)$

$= 110$ tenths of a second, 11 seconds

b) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

$$= 5(36 + x^2)^{\frac{1}{2}} + 80 - 4x$$

$$T'(x) = [\frac{5}{2}\left(36 + x^2\right)^{-\frac{1}{2}} \bullet 2x] - 4$$

$$= 5x(36 + x^2)^{-\frac{1}{2}} - 4$$

$$= \frac{5x}{(36 + x^2)^{\frac{1}{2}}} - 4$$

Minimum values occur where $T'(x) = 0$

Let $\frac{5x}{(36 + x^2)^{\frac{1}{2}}} - 4 = 0$

$$\frac{5x}{(36 + x^2)^{\frac{1}{2}}} = 4$$

$$4(36 + x^2)^{\frac{1}{2}} = 5x$$

$$(36 + x^2)^{\frac{1}{2}} = \frac{5x}{4}$$

Squaring both sides gives

$$36 + x^2 = \frac{25x^2}{16}$$

$$576 + 16x^2 = 25x^2$$

$$9x^2 - 576 = 0$$

$$x^2 - 64 = 0$$

$$x = 8, x = -8$$

Since x is a distance it cannot be negative so $x = 8$

We must prove that $x = 8$ does indeed give a minimum value of $T(x)$ using a nature table

x	1	8	10
$T'(x)$	$-$	0	$+$
Shape	\	—	/

So, $x = 8$ does give a minimum value of $T(x)$

$$T(8) = 5\sqrt{36 + 8^2} + 4(20 - 8)$$

$$= 98$$

So, the minimum time possible is 98 tenths of a second, 9.8 seconds

Question 9

Let $36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) = k\sin(1.5t - a)$

$$= k\sin\left(1.5t\right)\cos a - k\cos\left(1.5t\right)\sin a$$

$$= k\cos a\sin\left(1.5t\right) - k\sin a\cos(1.5t)$$

By inspection we have $36 = k\cos a$ and $15 = k\sin a$

$$k = \sqrt{36^2 + 15^2} = 39$$

$$\frac{k\sin a}{k\cos a} = \frac{15}{36} = \tan a$$

$a = tan^{-1}\left(\frac{15}{36}\right) = 0.39$ radians (confirm this answer is in the correct quadrant using CAST)

So, $36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) = 39\sin(1.5t - 0.39)$

$$h = 36\sin\left(1.5t\right) - 15\cos\left(1.5t\right) + 65$$

$= 39\sin(1.5t - 0.39) + 65$ (from above)

Let $h = 100$ to give

$$100 = 39\sin(1.5t - 0.39) + 65$$

$$35 = 39\sin(1.5t - 0.39)$$

$$\frac{35}{39} = \sin(1.5t - 0.39)$$

$1.5t - 0.39 = \sin^{-1}\left(\frac{35}{39}\right) = 1.11$ radians

$$1.5t = 1.5$$

$$t = 1$$

Second solution for $\frac{35}{39} = \sin(1.5t - 0.39)$ is given by

$\pi - 1.11 = 2.03$ radians, so

$1.5t - 0.39 = 2.03$

$t = 1.61$

So, the solutions are $t = 1$ second and $t = 1.61$ seconds