2014 Higher Paper 1

Question 1

$$u_{n+1} = \frac{1}{3}u_n + 1$$

$$u_3 = \frac{1}{3}u_2 + 1 = \frac{1}{3} \bullet 15 + 1 = 6$$

$$u_4 = \frac{1}{3}u_3 + 1 = \frac{1}{3} \bullet 6 + 1 = 3$$

Question 2

$$m_{CT} = \frac{2-(-1)}{1-3} = -\frac{3}{2}$$

$m_{tan} = \dfrac{2}{3}$ since perpendicular

Using $y - b = m(x - a)$ with $(3, -1)$ gives

$$y-(-1) = \frac{2}{3}(x-3)$$

$$y = \frac{2}{3}x - 3$$

Question 3

$$\log_4 12 - \log_4 x = \log_4 6$$

$$\log_4 12 - \log_4 6 = \log_4 x$$

$$\log_4 \frac{12}{6} = \log_4 x$$

$$\log_4 2 = \log_4 x$$

$$x = 2$$

Question 4

Let $3\sin x - 4\cos x = k\cos(x - a)$

$$= k\cos x\cos a + k\sin x\sin a$$

$$= k\cos a\cos x + k\sin a\sin x$$

By inspection $3 = k\sin a$ and $-4 = k\cos a$

Question 5

$$\int (2x+9)^5\,dx$$

$$= \frac{(2x+9)^6}{6 \bullet 2} + c$$

$$= \frac{(2x+9)^6}{12} + c$$

Question 6

$$2\underline{u} - 3\underline{v} = 2\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ 6 \end{pmatrix}$$

Question 7

$\sin 2a = 2\sin a\cos a$

From SOHCAHTOA

$$\sin 2a = 2 \bullet \frac{3}{\sqrt{34}} \bullet \frac{5}{\sqrt{34}} = \frac{30}{34} = \frac{15}{17}$$

Question 8

Let $f(x) = (4 - 9x^4)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(4 - 9x^4)^{-\frac{1}{2}} \bullet (-36x^3)$$

$$= -18x^3(4 - 9x^4)^{\frac{1}{2}}$$

Question 9

Note there are various ways to solve this problem. This approach uses the derivative to find max values.

Let $f(x) = 5\sin 2x + 5\sqrt{3}\cos 2x$

$$f'(x) = 10\cos 2x - 10\sin 2x$$

For maximum values $f'(x) = 0$

$$10\cos 2x - 10\sin 2x = 0$$

$$\cos 2x - \sin 2x = 0$$

$$\cos 2x = \sin 2x$$

$$1 = \frac{\sin 2x}{\cos 2x} = \tan 2x$$

From exact values $2x = \frac{\pi}{6}$

So, the maximum value occurs where $2x = \frac{\pi}{6}$

Substituting $2x = \frac{\pi}{6}$ gives

$$f(x) = 5\sin\frac{\pi}{6} + 5\sqrt{3}\cos\frac{\pi}{6} = \frac{5}{2} + \frac{15}{2} = 10$$

Question 10

For $u_{n+1} = au_n + b$ the limit occurs where $-1 < a < 1$

So, limit occurs where $-1 < (k-2) < 1$

Adding 2 to both sides gives $1 < k < 3$

Question 11

The co-ordinates given on $f(x)$ are $(2, 3)$ $(5, 0)$

Consider transforming these using $y = 2f(x) + 1$

This transformation tells us to multiply the y co-ordinate by 2 and add 1

So, $(2, 3)$ becomes $(2, 7)$

And $(5, 0)$ becomes (5, 1)

By inspection graph C has these points present

Question 12

$$f(x) = \frac{6x}{x^2 + 6x - 16}$$

Restrictions on the domain of $f(x)$ occur where $x^2 + 6x - 16 = 0$ since we cannot divide by 0

Let $x^2 + 6x - 16 = 0$

$(x + 8)(x - 2) = 0$

$x = -8, x = 2$

So, the restrictions on the domain of f are $x = -8, x = 2$

Question 13

$$\sin\frac{\pi}{3} - \cos\frac{5\pi}{4} = \frac{\sqrt{3}}{2} - (-\frac{1}{\sqrt{2}})$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} + \sqrt{2}}{2}$$

Question 14

Since $\underline{u}$ & $\underline{v}$ are perpendicular $\underline{u} \bullet \underline{v} = 0$

$$\underline{u} \bullet \underline{v} = \left(1 \bullet -6\right) + 2k + 5k = 0$$

$$7k - 6 = 0$$

$$k = \frac{6}{7}$$

Question 15

Since the roots are at $x = -1, x = 2$ we have $y = k(x+1)(x-2)^2$ for some value k

Substituting the point $(0, -8)$ gives

$$-8 = k(0+1)(0-2)^2$$

$$-8 = 4k$$

$$k = -2$$

So, $y = -2(x+1)(x-2)^2$

Question 16

$$\underline{a} \bullet \left(\underline{a} + 2\underline{b}\right) = \underline{a} \bullet \underline{a} + 2\underline{a} \bullet \underline{b}$$

$= \left|\underline{a}\right|\left|\underline{a}\right| cos\theta + 2 \bullet \frac{2}{3}$ where θ is the angle between $\underline{a}$ and itself, 0

$$= 1 \bullet 1 \bullet 1 + \frac{4}{3} = \frac{7}{3}$$

Question 17

$$3x^2 + 12x + 17 = 3\left(x^2 + 4x\right) + 17$$
$$= 3\left[(x+2)^2 - 4\right] + 17$$
$$= 3(x+2)^2 - 12 + 17$$
$$= 3(x+2)^2 + 5$$

Question 18

$1 - 2\sin^2 15° = \cos(2 \bullet 15°)$ from the double angle formulas

$$= \cos 30°$$
$$= \frac{\sqrt{3}}{2}$$

Question 19

Let C be the centre of the hexagon

$$\overrightarrow{SW} = \overrightarrow{SR} + \overrightarrow{RC} + \overrightarrow{CW}$$
$$= -\underline{u} - \underline{v} - \underline{v}$$
$$= -\underline{u} - 2\underline{v}$$

Question 20

$$2 - \log_5 \frac{1}{25} = 2 - (\log_5 1 - \log_5 25) = 2 - 0 + 2 = 4$$

Question 21

a) $y = 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2$$

For stationary points $\frac{dy}{dx} = 0$

$$6x - 3x^2 = 0$$
$$3x(2 - x) = 0$$

$x = 0, x = 2$

When $x = 0, y = 0$

When $x = 2, y = 4$

So, stationary points are $(0, 0)$ $(2, 4)$

x	-1	0	1	2	5
$f'(x)$	+	0	–	0	+
Shape	\	_	/	‾	\

$(0, 0)$ is a minimum turning point and $(2, 4)$ is a maximum turning point

b) For the y intercept $x = 0$ giving

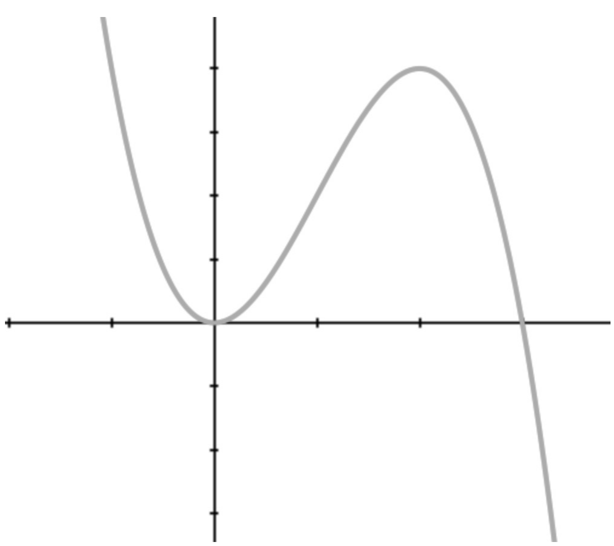
$y = 3 \bullet 0^2 - 0^3 = 0$

For the x intercept $y = 0$ giving

$0 = 3x^2 - x^3$

$0 = x^2(3 - x)$

$x = 0, x = 3$



Question 22

a) Let $f(x) = 6x^3 + 7x^2 + ax + b$

Since $x + 1$ is a factor $f(-1) = 0$

$f(-1) = -6 + 7 - a + b = 0$

Rearranging gives $a = b + 1 \qquad (1)$

Since 72 is the remainder when $f(x)$ is divided by $x - 2$ we have

$f(2) = 48 + 28 + 2a + b = 72$

Rearranging gives $2a = -b - 4 \qquad (2)$

Substituting (1) into (2) gives

$2(b + 1) = -b - 4$

$2b + 2 = -b - 4$

$3b = -6$

$b = -2$

Using (1) we have

$a = -2 + 1 = -1$

b) $f(x) = 6x^3 + 7x^2 - x - 2$

Since $x + 1$ is a factor we have

−1	6	7	-1	-2
		−6	-1	2
	6	1	-2	0

So, $f(x) = (x + 1)(6x^2 + x - 2)$

$= (x + 1)(3x + 2)(2x - 1)$

Question 23

a) $x^2 + y^2 + 2x - 4y - 15 = 0 \quad (1)$

Substitute $y = 3x - 5$ into (1) to give

$x^2 + (3x - 5)^2 + 2(3x - 5) - 4y - 15 = 0$

Simplifying gives

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1, x = 3$

When $x = 1, y = -2$

When $x = 3, y = -4$

$P = (1,\ -2),\ Q = (3,\ -4)$

b) Centre of $C_1 = (1,\ -2)$

$$m_{QT} = \frac{4 - 2}{3 - (-1)} = \frac{1}{2}$$

$$m_{PT} = \frac{2 - (-2)}{-1 - 1} = -2$$

Since $m_{QT} \bullet m_{PT} = -1$ QT and PT are perpendicular

c) Since triangle PTQ is right-angled (from B) PQ is a diameter of C_2

Midpoint of PQ $= (2, 1) =$ centre of C_2

Distance from (2, 1) to Q $= \sqrt{(4-2)^2 + (3-1)^2} = \sqrt{10} =$ radius of C_2

Equation of C_2 is $(x - 2)^2 + (y - 1)^2 = (\sqrt{10})^2$

$(x - 2)^2 + (y - 1)^2 = 10$

Question 24

$y = ka^x$

$\log_9 y = \log_9(ka^x)$

$\log_9 y = \log_9 k + \log_9 a^x$

$\log_9 y = \log_9 k + x\log_9 a$

Rewrite in the format of a straight line

$\log_9 y = (\log_9 a)x + \log_9 k$

Consider the line shown in the graph

$$m = \frac{5-2}{6-0} = \frac{1}{2}$$

Y intercept $= 2$

So, by inspection we have

$\log_9 a = \frac{1}{2}$ $\qquad$ $\log_9 k = 2$

$a = 9^{\frac{1}{2}} = 3$ $\qquad$ $k = 9^2 = 81$

Question 1

a) $m_{AB} = \frac{2-0}{5-3} = 1$

So, gradient of perpendicular line is -1

Midpoint of AB $= (4,\ 1)$

Using $y - b = m(x - a)$ with (4, 1) we have

$$y - 1 = -1(x - 4)$$

$$y = -x + 5 \quad (1)$$

b) $y + 2x = 6$

$$y = -2x + 6 \quad (2)$$

Equating (1) & (2) gives

$$-x + 5 = -2x + 6$$

$$x = 1$$

When $x = 1, y = 4$

So, $T = (1, 4)$

c) $A = (3, 0),\ T = (1, 4)$

$$m_{AT} = \frac{4-0}{1-3} = -2$$

Using $y - b = m(x - a)$ with (1, 4) gives

$$y - 4 = -2(x - 1)$$

Rearranging gives

$$y = -2x - 2$$

Question 2

$y = x^4 - 2x^3 + 5$

$\frac{dy}{dx} = 4x^3 - 6x^2$

When $x = 2$, $\frac{dy}{dx} = 32 - 24 = 8$

So, the gradient of the tangent at $x = 2$ is 8

When $x = 2$, $y = 2^4 - 2 \bullet 2^3 + 5 = 5$

Using $y - b = m(x - a)$ with $(2,5)$ gives

$y - 5 = 8(x - 2)$

Rearranging gives

$$y = 8x - 11$$

Question 3

a) $f\left(g(x)\right) = f(x + 3) = (x + 3)(x + 3 - 1) + q$

$$= (x + 3)(x + 2) + q$$

$$= x^2 + 5x + 6 + q$$

b) $f\left(g(x)\right) = x^2 + 5x + 6 + q = x^2 + 5x + (6 + q)$

$a = 1, b = 5, c = 6 + q$

For equal roots $b^2 - 4ac = 0$

$$25 - 4 \bullet 1 \bullet \left(6 + q\right) = 0$$

$$25 - 24 - 4q = 0$$

$$q = \frac{1}{4}$$

Question 4

a) $C = (11, 12, 6),\ D = (8, 8, 4)$

b) $\overrightarrow{CB} = \begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix} \quad \overrightarrow{CD} = \begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix}$

c) $\left|\overrightarrow{CB}\right| = \sqrt{0^2 + (-8)^2 + (-4)^2} = \sqrt{80}$

$$\left|\overrightarrow{CD}\right| = \sqrt{(-3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}$$

$$\overrightarrow{CB} \bullet \overrightarrow{CD} = (0 \bullet -3) + (-8 \bullet -4) + (-4 \bullet -2) = 40$$

$$\overrightarrow{CB} \bullet \overrightarrow{CD} = \left|\overrightarrow{CB}\right|\left|\overrightarrow{CD}\right| \cos(BCD)^\circ$$

$$40 = \sqrt{80}\sqrt{29}\cos(BCD)^\circ$$

$$\frac{40}{\sqrt{80}\sqrt{29}} = \cos(BCD)^\circ$$

$$BCD = cos^{-1}\left(\frac{40}{\sqrt{80}\sqrt{29}}\right)$$

$$BCD = 33.85^\circ$$

Question 5

$$\int_4^t (3x+4)^{-\frac{1}{2}}\,dx = 2$$

Using the reverse chain rule we have

$$\left[\frac{2}{3}(3x+4)^{\frac{1}{2}}\right]_4^t = 2$$

$$\frac{2}{3}(3t+4)^{\frac{1}{2}} - \frac{2}{3}(12+4)^{\frac{1}{2}} = 2$$

$$\frac{2}{3}(3t+4)^{\frac{1}{2}} - \frac{8}{3} = 2$$

Simplifying gives

$$(3t+4)^{\frac{1}{2}} = 7$$

$$3t+4 = 49,\ t = 15$$

Question 6

$$\sin x - 2\cos 2x = 1$$

Using the double angle formula $\cos 2x = 1 - 2\sin^2 x$ we have

$$\sin x - 2\left(1 - 2\sin^2 x\right) = 1$$

$$\sin x - 2 + 4\sin^2 x = 1$$

$$4\sin^2 x + \sin x - 3 = 0$$

$$(4\sin x - 3)(\sin x + 1) = 0$$

Separating into two equations gives

$$4\sin x - 3 = 0 \qquad\qquad \sin x + 1 = 0$$

$$\sin x = \frac{3}{4} \qquad\qquad \sin x = -1$$

$$x = 0.8\ rad \qquad\qquad x = \frac{3\pi}{2}$$

$$x = \pi - 0.8 = 2.3\ rad$$

Question 7

a) $y = 2x \qquad (1)$

$y = 6x - x^2 \qquad (2)$

For intersection equate (1) & (2) to give

$2x = 6x - x^2$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0, x = 4$

Area $= \int_0^4 \left[\left(6x - x^2\right) - 2x\right)] \, dx$

$= \int_0^4 \left(4x - x^2\right) dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left(32 - \frac{64}{3}\right) - (0)$

$= \frac{96}{3} - \frac{64}{3} = \frac{28}{3} \, units^2$

Area $= \frac{28}{3} \, x \, 300m^2 = 2{,}800m^2$

b) We must find the point of tangency of the line and the curve

Since the line is parallel to $y = 2x$ it has the same gradient, 2

So, let $\frac{d}{dx}\left(6x - x^2\right) = 2$ giving

$6 - 2x = 2$

$$x = 2$$

So, the point of tangency is at $x = 2$

When $x = 2$, $y = 6 \bullet 2 - 2^2 = 8$

So, the point of tangency is at (2, 8)

The equation of the 'road' line is given by $y = 2x + c$

To find c substitute (2, 8) which is a point on the line

$8 = 2 \bullet 2 + c$

$c = 4$

So $y = 2x + 4$

$$\text{Area } = \int_0^2 [(2x+4) - (6x - x^2)]\, dx$$

$$= \int_0^2 \left(x^2 - 4x + 4\right) dx$$

$$= [\frac{1}{3}x^3 - 2x^2 + 4x]_0^2$$

$$= \left(\frac{8}{3} - 4 - 8\right) - (0)$$

$$= \frac{20}{3}\, units^{\,2}$$

$$\text{Area } = \frac{20}{3} x\ 300 m^2 = 2{,}000\ m^2$$

Question 8

$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$

Radius $= \sqrt{p^2 + (2p)^2 - (3p + 2)}$

$= \sqrt{5p^2 - 3p - 2}$

Since the radius of a circle must be greater than zero we have

$\sqrt{5p^2 - 3p - 2} > 0$

$5p^2 - 3p - 2 > 0$

Let $5p^2 - 3p - 2 = 0$

$\left(5p + 2\right)\left(p - 1\right) = 0$

$p = -\frac{2}{5},\ p = 1$

So, for $5p^2 - 3p - 2 > 0$ we have

$1 < p < -\frac{2}{5}$

Question 9

a) $v(t) = 8\cos(2t - \frac{\pi}{2})$

$a(t) = v'(t) = -16\sin(2t - \frac{\pi}{2})$

b) $v'(10) = -16\sin\left(20 - \frac{\pi}{2}\right) = 6.53$

Since $6.53 > 0$ the velocity of the object is increasing when $t = 10$

c) $s(t) = \int v(t)\, dt$

$$= \int 8\cos(2t - \frac{\pi}{2})dt$$

$$= 4\sin\left(2t - \frac{\pi}{2}\right) + c$$

Since $s(t) = 4$ when $t = 0$ we have

$$4 = 4\sin(0 - \frac{\pi}{2}) + c$$

$$4 = 4\sin\left(-\frac{\pi}{2}\right) + c$$

$$4 = 4 \bullet (-1) + c$$

$$c = 8$$

So, $s(t) = 4\sin\left(2t - \frac{\pi}{2}\right) + 8$