2013 Higher Paper 1

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Paper 1: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Paper 2: 1 2 3 4 5 6 7 8 9

Question 1

$$g\left(f(x)\right) = g\left(x^2+1\right) = 3\left(x^2+1\right) - 4$$
$$= 3x^2 + 3 - 4$$
$$= 3x^2 - 1$$

Question 2

$$y = x^2 - 4x + 7$$
$$\frac{dy}{dx} = 2x - 4$$

Gradient of tangent at (5, 12) $= 2 \bullet 5 - 4 = 6$

Using $y - b = m(x - a)$ gives

$$y - 12 = 6(x - 5)$$
$$y - 12 = 6x - 30$$
$$y = 6x - 28$$

Question 3

$$a = 2, b = 4, c = 5$$
$$b^2 - 4ac = 4^2 - 4 \bullet 2 \bullet 5$$
$$= 16 - 40$$
$$= -24$$

Question 4

The graph of this function has an amplitude of 4, a period of 180°, and is shifted down by 1 unit.

So, the solution is A.

Question 5

$5x + 3y - 6 = 0$

Rearranging gives

$y = -\frac{5}{3}x + 2$

Gradient of this line is $-\frac{5}{3}$

Since L is parallel it also has gradient $-\frac{5}{3}$

Using $y - b = m(x - a)$ with $(-2, -1)$ gives

$$y - (-1) = -\frac{5}{3}(x - (-2))$$

Rearranging gives

$$y = -\frac{5}{3}x - \frac{8}{3}$$

Question 6

Let $f(x) = x^3 + 3x^2 - 5x - 6$

$f(2) = 2^3 + 3 \bullet 2^2 - 5 \bullet 2 - 6 = 4$

The remainder is 4 when $f(x)$ is divided by $(x - 2)$

Question 7

$\int x(3x + 2)\, dx$

$= \int \left(3x^2 + 2x\right)\, dx$

$= x^3 + x^2 + c$

Question 8

$$u_{n+1} = 0.1u_n + 8$$
$$u_1 = 0.1u_0 + 8$$
$$11 = 0.1u_0 + 8$$
$$3 = 0.1u_0$$
$$30 = u_0$$

Since $-1 < 0.1 < 1$ the sequence does have a limit as n $\to \infty$

The answer is C) Only statement 2) is correct

Question 9

$$\sin 2x = 2\sin x \cos x$$
$$= 2 \bullet \frac{1}{\sqrt{5}} \bullet \frac{2}{\sqrt{5}}$$
$$= \frac{4}{5}$$

Question 10

$$\cos(270 - a) = \cos 270 \cos a + \sin 270 \sin a$$
$$= 0 \bullet \cos a + (-1) \bullet \sin a$$
$$= -\sin a$$

Question 11

$$y = -f(x - k)$$

Compared to the graph of $y = f(x)$ this graph is inverted and shifted ‘k’ units to the right

By inspection B is the correct solution

Question 12

$$\underline{f} + \underline{g} = \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix}$$

$$\left| \underline{f} + \underline{g} \right| = \sqrt{5^2 + 4^2 + 5^2}$$

$$= \sqrt{64}$$

$$= 8$$

Question 13

Let $x^2 - 7x + 12 = 0$

$$(x - 4)(x - 3) = 0$$

$$x = 3, x = 4$$

So $x = 3, x = 4$ cannot be in the domain of $f(x)$

Question 14

$$\underline{a} \,.\, (\underline{a} + \underline{b}) = \underline{a} \,.\, \underline{a} + \underline{a} \,.\, \underline{b}$$

$= \left|\underline{a}\right|\left|\underline{a}\right| cos\theta + 5$ where θ is the angle between $\underline{a}$ and itself

$$= 3 \bullet 3 \bullet \cos 0 + 5$$

$$= 9 + 5$$

$$= 14$$

Question 15

$\tan\frac{\pi}{4} = 1$ so $\frac{\pi}{4}$ is our reference angle

Using CAST tan is negative in quadrants 2 & 4

So, the solutions are $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

However, these solutions are for $\frac{x}{2}$ giving $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$

But $\frac{5\pi}{2}$ is out of our range so $x = \frac{3\pi}{2}$

Question 16

$$\int (1-6x)^{-\frac{1}{2}}\,dx$$

$$= \frac{(1-6x)^{\frac{1}{2}}}{\frac{1}{2} \bullet (-6)} + c$$

$$= -\frac{1}{3}\left(1-6x\right)^{\frac{1}{2}} + c$$

Question 17

$y = kx(x+a)^2$

Rewriting this gives

$y = k(x-0)(x+a)(x+a)$

This shows that the roots are at $x = 0, x = -a$

So, by inspection, $a = 2$

So $y = kx(x+2)^2$

Substituting the point (1, 3) gives

$3 = k(1+2)^2$

$3 = 9k$

$k = \frac{1}{3}$

Question 18

$y = \sin(x^2 - 3)$

$\frac{dy}{dx} = \cos(x^2 - 3) \bullet 2x$

$= 2x\cos(x^2 - 3)$

Question 19

$1 - 2x - 3x^2 > 0$

Multiplying through by -1 gives

$3x^2 + 2x - 1 < 0$

$(3x - 1)(x + 1) < 0$

Consider $(3x - 1)(x + 1) = 0$

$x = -1, x = \frac{1}{3}$

These are the roots of the quadratic. By considering the graph we see

It is negative (i.e. $y < 0$) for $-1 < x < \frac{1}{3}$

The solutions are all x such that $-1 < x < \frac{1}{3}$

Question 20

The equation of the line is $\log_3 y = 2x$ since gradient $= 2$ and y-intercept $= 0$

$\log_3 y = 2x$

Rewriting as an exponential gives

$y = 3^{2x}$

Question 21

$2x^2 + 12x + 1$

$= 2(x^2 + 6x) + 1$

Now complete the square on the inside of the bracket

$= 2\left[(x + 3)^2 - 9\right] + 1$

$= 2(x + 3)^2 - 18 + 1$

$= 2(x + 3)^2 - 17$

Question 22

$x^2 + y^2 + 2x + 4y - 27 = 0$

a) Centre $= (-1, -2)$, Radius $= \sqrt{1^2 + 2^2 + 27} = \sqrt{32}$

b) Let C be the centre of C_1

$m_{cp} = 1$

Therefore $m_{tan} = -1$ since perpendicular gradients

Using $y - b = m(x - a)$ with (3, 2) gives

$y - 2 = -1(x - 2)$

$y = -x + 5$

c) Radius of $c_2 = \dfrac{\sqrt{32}}{2} = \sqrt{8}$

Since the centre of $c_2 = (10, -1)$ we have

$(x - 10)^2 + (y + 1)^2 = (\sqrt{8})^2$

$(x - 10)^2 + (y + 1)^2 = 8$

Expanding brackets and collecting terms gives

$x^2 + y^2 - 20x + 2y + 93 = 0$

d) Substituting $y = -x + 5$ into c_2 gives

$x^2 + (5 - x)^2 - 20(5 - x) + 2y + 93 = 0$

Expanding and simplifying gives

$x^2 - 16x + 64 = 0$

Using the discriminant with $a = 1$, $b = -16$, $c = 64$ gives

$b^2 - 4ac = \left(-16\right)^2 - 4 \bullet 1 \bullet 64 = 0$

Since the discriminant equals 0 there is only one solution

Therefore, there is only one point at which the line $y = -x + 5$

Intersects with c_2. In other words, it is a tangent

Question 23

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x - a) = k\sin x\cos a - k\cos x\sin a$

$$= k\cos a\sin x - k\sin a\cos x$$

By inspection we have $\sqrt{3} = k\cos a$ and $1 = k\sin a$

$$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\frac{k\sin a}{k\cos a} = \frac{1}{\sqrt{3}} = \tan a$$

Giving $a = 30°$ by exact values

Since we want a to be such that both $\sin a$ and $\cos a$ are positive, by CAST,

a must be in quadrant 1. So, $30°$ is correct

Thus $\sqrt{3}\sin x - \cos x = 2\sin(x - 30)°$

b) $4 + 5\cos x - 5\sqrt{3}\sin x = 4 - 5(\sqrt{3}\sin x - \cos x)$

$= 4 - 5 \bullet 2\sin(x - 30)$ from a)

$= 4 - 10\sin(x - 30)$

$= -10\sin(x - 30) + 4$

$-10\sin(x - 30)$ has a maximum value of 10

So $= -10\sin(x - 30) + 4$ has a maximum value of 14

Question 24

a) i) $\overrightarrow{AT} = \begin{pmatrix} 10 \\ 10 \\ 4 \end{pmatrix}$ $\quad \overrightarrow{TB} = \begin{pmatrix} 15 \\ 15 \\ 6 \end{pmatrix}$

$$\overrightarrow{AT} = 2\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} \quad \overrightarrow{TB} = 3\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$$

$$\frac{1}{2}\overrightarrow{AT} = \frac{1}{3}\overrightarrow{TB}$$

$$3\overrightarrow{AT} = 2\overrightarrow{TB}$$

This shows that $\overrightarrow{AT}$ & $\overrightarrow{TB}$ are parallel but since T is a common point,

It follows that $A,\ T\ \ B$ are collinear.

ii) The ratio is 2 : 3

b) Since C lies on the x-axis it has a y co-ordinate of 0 and a z co-ordinate of 0.

Let $C = (c,\ 0, 0)$

Then $\overrightarrow{TC} = C - T = \begin{pmatrix} c-3 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{TB} \bullet \overrightarrow{TC} = 0$ since they are perpendicular

$$\overrightarrow{TB} \bullet \overrightarrow{TC} = 15(c-3) + \left(15 \bullet 0\right) + \left(6 \bullet 0\right) = 0$$

$$15c - 45 = 0$$

$$c = 3$$

So, $c = (3, 0, 0)$

2013 Higher Paper 2

Question 1

$u_1 = 4, u_2 = 7, u_3 = 16$

$u_{n+1} = mu_n + c$

Substituting u_1 and u_2 gives

$u_2 = mu_1 + c$

$7 = 4m + c$

Substituting u_2 and u_3 gives

$u_3 = mu_2 + c$

$16 = 7m + c$

Now we have simultaneous equations to solve for m c

$7 = 4m + c$ (1)

$16 = 7m + c$ (2)

(2) - (1) gives

$9 = 3m$

$m = 3$

Substituting $m = 3$ into (2) gives

$16 = 7 \bullet 3 + c$

$c = -5$

So, $m = 3, c = -5$

Question 2

a) $m_{PQ} = \dfrac{6-2}{5-7} = -2$

$m_{QR} = \dfrac{1}{2}$ since PQ QR are perpendicular

Using $y - b = m(x - a)$ with $Q = (5, 6)$ we have

$$y - 6 = \frac{1}{2}(x - 5)$$

$$y = \frac{1}{2}x + \frac{7}{2}$$

b) $x + 3y = 13$

$$y = -\frac{1}{3}x + \frac{13}{3}$$

Equating QR with PT gives

$$\frac{1}{2}x + \frac{7}{2} = -\frac{1}{3}x + \frac{13}{3}$$

Multiply through by 6 to simplify

$$3x + 21 = -2x + 26$$

$$x = 1$$

Substituting $x = 1$ into $y = \frac{1}{2}x + \frac{7}{2}$ gives

$$y = \frac{1}{2} + \frac{7}{2} = 4$$

$T = (1, 4)$

c) $\overrightarrow{QT} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

So, $R = (1 - 4, 4 - 2) = (-3, 2)$

$\overrightarrow{QP} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

So, $S = (-3 + 2, 2 - 4) = (-1, -2)$

Question 3

a) Let $f(x) = x^3 + 3x^2 + x - 5$

Using synthetic division gives

1	1	3	1	-5
		1	4	5
	1	4	5	0

$f(x) = (x-1)(x^2 + 4x + 5)$

Note that $x^2 + 4x + 5$ does not factorise

b) Let $g(x) = x^4 + 4x^3 + 2x^2 - 20x + 3$

$g'(x) = 4x^3 + 12x^2 + 4x - 20$

Stationary Points occur where $g'(x) = 0$ giving

$$4x^3 + 12x^2 + 4x - 20 = 0$$
$$x^3 + 3x^2 + x - 5 = 0$$
$$(x-1)\left(x^2 + 4x + 5\right) = 0 \text{ from a)}$$

The solution to this equation is $x = 1$ since $x^2 + 4x + 5$ has no solutions $(b^2 - 4ac = -4 < 0)$

Thus $g(x)$ has only one stationary point

Question 4

Equation the line and curves equations gives

$2x + 3 = x^3 + 3x^2 + 2x + 3$

$x^3 + 3x^2 = 0$

$x^2(x+3) = 0$

$x = 0, x = -3$

When $x = -3, y = 2 \bullet (-3) + 3 = -3$

So $B = (-3, -3)$

$$\text{Area} = \int_{-3}^{0} (x^3 + 3x^2 + 2x + 3) - (2x + 3)\, dx$$

$$= \int_{-3}^{0} (x^3 + 3x^2) dx$$

$$= \left[\frac{x^4}{4} + x^3\right]_{-3}^{0}$$

$$= (0) - \left(\frac{81}{4} - 27\right) = \frac{27}{4} units^2$$

Question 5

$$\log_5(3 - 2x) + \log_5(2 + x) = 1$$

$$\log_5\left[(3 - 2x)(2 + x)\right] = 1$$

$$5^1 = (3 - 2x)(2 + x)$$

$$5 = 6 - x - 2x^2$$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$$x = -1, x = \frac{1}{2}$$

Question 6

$$\int_0^a 5\sin 3x\, dx = \frac{10}{3}$$

$$\left[-\frac{5}{3}\cos 3x\right]_0^a = \frac{10}{3}$$

$$(-\frac{5}{3}\cos 3a) - (-\frac{5}{3}\cos 0) = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a + \frac{5}{3} = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a = \frac{5}{3}$$

$$\cos 3a = -1$$

$$3a = \pi,\ a = \frac{\pi}{3}$$

There are no other solutions in $0 \le a < \pi$

Question 7

a) $L = 3x + 4y$ by inspection of the diagram

Area $= 24 = 2xy$

$$y = \frac{24}{2x} = \frac{12}{x}$$

Substituting $y = \frac{12}{x}$ into L gives

$$L = 3x + 4 \bullet \frac{12}{x} = 3x + \frac{48}{x}$$

b) i) $L(x) = 3x + \frac{48}{x}$

$$= 3x + 48x^{-1}$$

$$L'(x) = 3 - 48x^{-2}$$

$$= 3 - \frac{48}{x^2}$$

Minimum values occur where $L'(x) = 0$

$$3 - \frac{48}{x^2} = 0$$

$$3x^2 - 48 = 0$$

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

Since x is a length it cannot be negative, so $x = 4$

x	1	4	10
$L'(x)$	+	0	−
Shape	\	—	/

$x = 4$ gives a minimum value of $L(x)$

ii) Minimum length occurs where $x = 4$

$$L(4) = 3 \bullet 4 + \frac{48}{4} = 24m$$

Minimum cost $= 24 \bullet £8.25 = £198$

Question 8

$\sin 2x = 2\cos^2 x$

$2\sin x\cos x = 2\cos^2 x$

$2\sin x\cos x - 2\cos^2 x = 0$

$2\cos x(\sin x - \cos x) = 0$

Separating into two equations gives

$2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x - \cos x = 0$

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Question 9

a) $p_t = p_0 e^{-kt}$

Since the concentration has halved $p_t = \frac{p_0}{2}$

$\frac{p_0}{2} = p_0 e^{-25k}$

$0.5 = e^{-25k}$

$\log_e 0.5 = \log_e e^{-25k}$

$\log_e 0.5 = -25k$

$k = \frac{\log_e 0.5}{-25}$

$k = 0.028$ to 2 significant figures

b) $p_t = p_0 e^{-kt}$

Let $t = 80$ and $k = 0.028$. Substitute to give

$p_t = p_0 e^{-0.028 \; x \; 80}$

$p_t = p_0 e^{-2.24}$

$p_t = 0.1065 \; p_0$

Rounding gives

$p_t = 0.11 \; p_0$

So, p_t is 11% of p_0

Therefore, the concentration has decreased by 89%