Higher Mathematics

Integration - **Solutions** - 2013-2017

Marks are indicated in brackets after each question number

2013 Paper 1 Question 7, (2)

$$\int x(3x+2)\,dx$$

$$= \int \left(3x^2+2x\right)\,dx$$

$$= x^3 + x^2 + c$$

2013 Paper 1 Question 16, (2)

$$\int (1-6x)^{-\frac{1}{2}}\,dx$$

$$= \frac{(1-6x)^{\frac{1}{2}}}{\frac{1}{2}\bullet(-6)} + c$$

$$= -\frac{1}{3}\left(1-6x\right)^{\frac{1}{2}} + c$$

2013 Paper 2 Question 4, (6)

Equation the line and curves equations gives

$$2x+3 = x^3+3x^2+2x+3$$

$$x^3+3x^2 = 0$$

$$x^2(x+3) = 0$$

$$x = 0,\, x = -3$$

When $x = -3,\, y = 2\bullet(-3)+3 = -3$

So $B = (-3,\ -3)$

$$\text{Area} = \int_{-3}^{0} (x^3 + 3x^2 + 2x + 3) - (2x + 3)\, dx$$

$$= \int_{-3}^{0} (x^3 + 3x^2) dx$$

$$= \left[\frac{x^4}{4} + x^3 \right]_{-3}^{0}$$

$$= (0) - \left(\frac{81}{4} - 27 \right) = \frac{27}{4} units^2$$

<u>2013 Paper 2 Question 6, (5)</u>

$$\int_0^a 5\sin 3x\, dx = \frac{10}{3}$$

$$\left[-\frac{5}{3}\cos 3x \right]_0^a = \frac{10}{3}$$

$$(-\frac{5}{3}\cos 3a) - (-\frac{5}{3}\cos 0) = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a + \frac{5}{3} = \frac{10}{3}$$

$$-\frac{5}{3}\cos 3a = \frac{5}{3}$$

$$\cos 3a = -1$$

$$3a = \pi,\ a = \frac{\pi}{3}$$

There are no other solutions in $0 \leq a < \pi$

2014 Paper 1 Question 5, (2)

$$\int (2x+9)^5 \, dx$$

$$= \frac{(2x+9)^6}{6 \bullet 2} + c$$

$$= \frac{(2x+9)^6}{12} + c$$

2014 Paper 2 Question 5, (5)

$$\int_4^t (3x+4)^{-\frac{1}{2}} \, dx = 2$$

Using the reverse chain rule we have

$$\left[\frac{2}{3}(3x+4)^{\frac{1}{2}}\right]_4^t = 2$$

$$\frac{2}{3}(3t+4)^{\frac{1}{2}} - \frac{2}{3}(12+4)^{\frac{1}{2}} = 2$$

$$\frac{2}{3}(3t+4)^{\frac{1}{2}} - \frac{8}{3} = 2$$

Simplifying gives

$$(3t+4)^{\frac{1}{2}} = 7$$

$$3t+4 = 49$$

$$t = 15$$

2014 Paper 2 Question 7, (5) (5)

a) $y = 2x \qquad (1)$

$y = 6x - x^2 \qquad (2)$

For intersection equate (1) & (2)

$$2x = 6x - x^2$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0, x = 4$$

Area $= \int_0^4 \left[\left(6x - x^2\right) - 2x\right)]\, dx$

$$= \int_0^4 \left(4x - x^2\right)\, dx$$

$$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$$

$$= \left(32 - \frac{64}{3}\right) - (0)$$

$$= \frac{96}{3} - \frac{64}{3} = \frac{28}{3}\, units^2$$

Area $= \frac{28}{3}\, x\ 300m^2 = 2{,}800m^2$

b) We must find the point of tangency of the line and the curve

Since the line is parallel to $y = 2x$ it has the same gradient, 2

So, let $\frac{d}{dx}(6x - x^2) = 2$ giving

$$6 - 2x = 2$$

$$x = 2$$

So, the point of tangency is at $x = 2$

When $x = 2$, $y = 6 \bullet 2 - 2^2 = 8$

So, the point of tangency is at (2, 8)

The equation of the 'road' line is given by $y = 2x + c$

To find c substitute (2, 8) which is a point on the line

$8 = 2 \bullet 2 + c$

$c = 4$

So $y = 2x + 4$

$$\text{Area } = \int_0^2 [(2x + 4) - (6x - x^2)]\, dx$$

$$= \int_0^2 (x^2 - 4x + 4)\, dx$$

$$= [\frac{1}{3}x^3 - 2x^2 + 4x]_0^2$$

$$= \left(\frac{8}{3} - 4 - 8\right) - (0)$$

$$= \frac{20}{3}\, units^2$$

$$\text{Area } = \frac{20}{3} x\ 300m^2 = 2{,}000\ m^2$$

2015 Paper 1 Question 12, (4)

$$\int_0^{\frac{3\pi}{4}} (a\cos bx)\; dx = \int_0^{\frac{\pi}{4}} (a\cos bx)\; dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (a\cos bx)\; dx$$

$$= \frac{1}{2} - (2 \bullet \frac{1}{2})$$

Note that since the part of the graph between $\frac{\pi}{4}$ $\frac{3\pi}{4}$ is under the x axis it has a negative integral

$$= -\frac{1}{2}$$

2015 Paper 2 Question 4, (2) (7)

a) Equating $f(x)$ $g(x)$ gives

$$\frac{1}{4}x^2 - \frac{1}{2}x + 3 = \frac{1}{4}x^2 - \frac{3}{2}x + 5$$

Multiplying through by 4 gives

$$x^2 - 2x + 12 = x^2 - 6x + 20$$

$$4x = 8, x = 2$$

b) Area of half of the plaque is given by

$$\int_0^2 f(x) - h(x)\; dx$$

$$= \int_0^2 (\frac{1}{4}x^2 - \frac{1}{2}x + 3) - (\frac{3}{8}x^2 - \frac{9}{4}x + 3)\; dx$$

$$= \int_0^2 \left(-\frac{1}{8}x^2 + \frac{7}{4}x\right) dx$$

$$= \left[-\frac{1}{24}x^3 + \frac{7}{8}x^2\right]_0^2$$

$$= \left(-\frac{8}{24} + \frac{28}{8}\right) - (0)$$

$= \frac{19}{6}$ after simplifying

Multiply by 2 to get the total area $= \frac{19}{2} units^2$

2015 Paper 2 Question 7, (2) (2) (2)

a) $\int (3\cos 2x + 1)\, dx = \frac{3}{2}\sin 2x + x + c$

b) $3\cos 2x + 1 = 3\left(\cos^2 x - \sin^2 x\right) + 1$

$$= 3\cos^2 x - 3\sin^2 x + 1$$

Using $\cos^2 x + \sin^2 x = 1$ to give

$$= 3\cos^2 x - 3\sin^2 x + \cos^2 x + \sin^2 x$$

$$= 4\cos^2 x - 2\sin^2 x$$

c) $\int (\sin^2 x - 2\cos^2 x)\, dx$

$$\sin^2 x - 2\cos^2 x = -\frac{1}{2}(4\cos^2 x - 2\sin^2 x)$$

$$= -\frac{1}{2}(3\cos 2x + 1)$$

$$= -\frac{3}{2}\cos 2x - \frac{1}{2}$$

So, $\int (\sin^2 x - 2\cos^2 x)\, dx = \int \left(-\frac{3}{2}\cos 2x - \frac{1}{2}\right) dx$

$$= -\frac{3}{4}\sin 2x - \frac{1}{2}x + c$$

2016 Paper 1 Question 5, (2)

$$\int 8\cos(4x+1)\,dx$$

$$= \frac{8\sin(4x+1)}{4} + c$$

$$= 2\sin(4x+1) + c$$

2016 Paper 2 Question 3, (2) (3) (1) (4)

a) i) Let $f(x) = 2x^3 - 9x^2 + 3x + 14$

Using synthetic division gives

-1	2	-9	3	14
		-2	11	-14
	2	-11	14	0

Since the remainder is zero $(x+1)$ is a factor of $f(x)$

ii) $2x^3 - 9x^2 + 3x + 14 = 0$

$(x+1)(2x^2 - 11x + 14) = 0$

$(x+1)(2x-7)(x-2) = 0$

$x = -1, x = 2, x = \frac{7}{2}$

b) i) $y = 2x^3 - 9x^2 + 3x + 14$

$y = (x+1)(2x-7)(x-2)$

By inspection of the roots $A = (-1, 0),\ B = (2, 0)$

ii) Area $= \int_{-1}^{2} \left(2x^3 - 9x^2 + 3x + 14\right)\, dx$

$= [\frac{2}{4}x^4 - \frac{9}{3}x^3 + \frac{3}{2}x^2 + 14x]_{-1}^{2}$

$= [\frac{1}{2}x^4 - 3x^3 + \frac{3}{2}x^2 + 14x]_{-1}^{2}$

$= \left(8 - 24 + 6 + 28\right) - (\frac{1}{2} + 3 + \frac{3}{2} - 14)$

$= 27$

Area $= 27\ units^2$

<u>2016 Paper 2 Question 9, (4)</u>

$f'(x) = \dfrac{2x + 1}{\sqrt{x}}$

$f(x) = \int f'(x)\, dx$

$= \int \dfrac{2x + 1}{\sqrt{x}}\, dx$

$= \int \dfrac{2x + 1}{x^{\frac{1}{2}}}\, dx$

$= \int (2x^{\frac{1}{2}} + x^{-\frac{1}{2}})\, dx$

$= \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

Since $f(9) = 40$ we have

$40 = \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

$40 = 36 + 6 + c$

$$c = -2$$

So, $f(x) = \dfrac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2$

<u>2017 Paper 1 Question 10, (5) (4)</u>

a) Shaded Area $= \int_0^2 \left(top\, function - bottom\, function\right) dx$

$$= \int_0^2 \left(x^3 - 4x^2 + 3x + 1\right) - \left(x^2 - 3x + 1\right) dx$$

$$= \int_0^2 (x^3 - 5x^2 + 6)\, dx$$

$$= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 6x]_0^2$$

$$= \left(4 - \frac{40}{3} + 12\right) - (0)$$

$$= \frac{8}{3}\, units^2$$

b) Area $= \int_0^2 \left(top\, function - bottom\, function\right) dx$

$$= \int_0^2 (1 - x) - \left(x^2 - 3x + 1\right) dx$$

$$= \int_0^2 \left(2x - x^2\right) dx$$

$$= [x^2 - \frac{1}{3}x^3]_0^2$$

$$= \left(4 - \frac{8}{3}\right) - (0)$$

$= \frac{4}{3}\, units^2$

Since the total shaded area is $\frac{8}{3}\, units^2$ $\frac{1}{2}$ of the shaded area is below the line $1 - x$

2017 Paper 1 Question 13, (4)

$$\int \frac{1}{(5-4x)^{\frac{1}{2}}}\, dx$$

$$= \int (5-4x)^{-\frac{1}{2}}\, dx$$

$$= \frac{(5-4x)^{\frac{1}{2}}}{\frac{1}{2} \bullet (-4)} + c$$

$$= -\frac{1}{8}\left(5-4x\right)^{\frac{1}{2}} + c$$