Higher Mathematics

Circle - **Solutions** - 2013-2017

Marks are indicated in brackets after each question

2013 Paper 1 Question 22, (2) (3) (3) (4)

$x^2 + y^2 + 2x + 4y - 27 = 0$

a) Centre $= (-1, -2)$, Radius $= \sqrt{1^2 + 2^2 + 27} = \sqrt{32}$

b) Let C be the centre of C_1

$m_{cp} = 1$

Therefore $m_{tan} = -1$ since perpendicular gradients

Using $y - b = m(x - a)$ with (3, 2) gives

$y - 2 = -1(x - 2)$

$y = -x + 5$

c) Radius of $c_2 = \dfrac{\sqrt{32}}{2} = \sqrt{8}$

Since the centre of $c_2 = (10, -1)$ we have

$(x - 10)^2 + (y + 1)^2 = (\sqrt{8})^2$

$(x - 10)^2 + (y + 1)^2 = 8$

Expanding brackets and collecting terms gives

$x^2 + y^2 - 20x + 2y + 93 = 0$

d) Substituting $y = -x + 5$ into c_2 gives

$x^2 + (5 - x)^2 - 20(5 - x) + 2y + 93 = 0$

Expanding and simplifying gives

$x^2 - 16x + 64 = 0$

Using the discriminant with $a = 1, b = -16, c = 64$ gives

$$b^2 - 4ac = \left(-16\right)^2 - 4 \bullet 1 \bullet 64 = 0$$

Since the discriminant equals 0 there is only one solution

Therefore, there is only one point at which the line $y = -x + 5$

Intersects with c_2. In other words, it is a tangent

<u>2014 Paper 1 Question 2, (2)</u>

$$m_{CT} = \frac{2 - (-1)}{1 - 3} = -\frac{3}{2}$$

$m_{tan} = \frac{2}{3}$ since perpendicular

Using $y - b = m(x - a)$ with $(3, -1)$ gives

$$y - (-1) = \frac{2}{3}(x - 3)$$

$$y = \frac{2}{3}x - 3$$

<u>2014 Paper 1 Question 23, (4) (3) (3)</u>

a) $x^2 + y^2 + 2x - 4y - 15 = 0 \quad (1)$

Substitute $y = 3x - 5$ into (1)

$$x^2 + (3x - 5)^2 + 2(3x - 5) - 4y - 15 = 0$$

Simplifying gives

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$$x = 1, x = 3$$

When $x = 1, y = -2$

When $x = 3, y = -4$

$$P = (1, -2),\ Q = (3, -4)$$

b) Centre of $C_1 = (1, -2)$

$$m_{QT} = \frac{4-2}{3-(-1)} = \frac{1}{2}$$

$$m_{PT} = \frac{2-(-2)}{-1-1} = -2$$

Since $m_{QT} \bullet m_{PT} = -1$ QT and PT are perpendicular

c) Since triangle PTQ is right-angled (from B) PQ is a diameter of C_2

Midpoint of PQ $= (2, 1) =$ centre of C_2

Distance from (2, 1) to Q $= \sqrt{(4-2)^2 + (3-1)^2} = \sqrt{10} =$ radius of C_2

Equation of C_2 is $(x-2)^2 + (y-1)^2 = (\sqrt{10})^2$

$$(x-2)^2 + (y-1)^2 = 10$$

<u>2014 Paper 2 Question 8, (5)</u>

$$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$$

Radius $= \sqrt{p^2 + (2p)^2 - (3p+2)}$

$= \sqrt{5p^2 - 3p - 2}$

Since the radius of a circle must be greater than zero we have

$$\sqrt{5p^2 - 3p - 2} > 0$$

$$5p^2 - 3p - 2 > 0$$

Let $5p^2 - 3p - 2 = 0$

$$(5p+2)(p-1) = 0$$

$$p = -\frac{2}{5},\ p = 1$$

So, for $5p^2 - 3p - 2 > 0$ we have

$$1 < p < -\frac{2}{5}$$

<u>2015 Paper 1 Question 11, (4) (6)</u>

a) Circle centre $= (-8, -2)$, radius $= \sqrt{45}$

Let circle centre be C

$$m_{CT} = \frac{-2-(-5)}{-8-(-2)} = -\frac{1}{2}$$

$m_{tan} = 2$ since the tangent is perpendicular to CT

Using $y - b = m(x - a)$ with $(-2, -5)$ gives

$$y - (-5) = 2(x - (-2))$$

$$y + 5 = 2x + 4$$

$$y = 2x - 1$$

b) Consider where the line, $y = 2x - 1$, and the parabola meet

To find this point equate them to give

$$2x - 1 = -2x^2 + px + 1 - p$$

$$2x^2 + 2x - px + p - 2 = 0$$

Grouping terms gives

$$2x^2 + (2 - p)x + (p - 2) = 0$$

The solutions to this equation are the points where the line and parabola meet.

Since the line is a tangent to the parabola they meet at only one point.

$$a = 2, b = (2 - p),\ c = (p - 2)$$

Since there is one solution $b^2 - 4ac = 0$

$$b^2 - 4ac = (2 - p)^2 - 4 \bullet 2 \bullet (p - 2) = 0$$

$$4 - 4p + p^2 - 8p + 16 = 0$$
$$p^2 - 12p + 20 = 0$$
$$(p - 10)(p - 2) = 0$$
$$p = 2, p = 10$$

But since $p > 3$ the solution is $p > 10$

<u>2015 Paper 1 Question 14, (2)</u>

$x^2 + y^2 - 12x - 10y + k = 0$

Centre $= (6, 5)$ and Radius $= \sqrt{(-6)^2 + (-5)^2 - k}$

$$= \sqrt{36 + 25 - k}$$
$$= \sqrt{51 - k}$$

Mark the centre of the circle on an x-y axis. If the circle meets the axes at exactly three points it is soon seen that the radius $= 6$

So, $\sqrt{51 - k} = 6$

$$51 - k = 36$$
$$k = 25$$

<u>2015 Paper 2 Question 5, (4) (4)</u>

a) Centre of $C_1 = (-3, -5)$

Centre of $C_2 = (9, 11)$

Distance between C_1 $C_2 = \sqrt{(11 - (-5))^2 + (9 - (-3))^2} = 20$

Radius of $C_1 = \sqrt{3^2 + 5^2 - 9} = 5$

Since the distance between the circle centres is 20 and the radius of C_1 is 5, the radius of $C_2 = 15$

b) The diameter of C_3 is equal to the diameter of $C_1 + C_2 = 40$. So, the radius of $C_3 = 20$

The centre of C_3 lies along the same line as the centres of C_1 C_2 since they are collinear

By considering the radii of the circles we see that the centre of C_3 lies $\frac{3}{4}$ of the way along the line from the centre of C_1 to C_2 (use a sketch to confirm this)

The distance between the x co-ordinate of C_1 C_2 is 12; $\frac{3}{4}$ of 12 $= 9$. So, starting at -3 and moving units gives 6.

The distance between the y co-ordinate of C_1 C_2 is 16; $\frac{3}{4}$ of 16 $= 12$. So, starting at -5 and moving 12 units gives 7.

So, the centre of $C_3 = \left(6, 7\right)$

Giving the equation of C_3 is $(x-6)^2 + (y-7)^2 = 20^2$

<u>2016 Paper 1 Question 4, (3)</u>

Diameter = distance between A & B

$$= \sqrt{(5-3)^2 + (1-(-7))^2}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Radius $= \frac{2\sqrt{17}}{2} = \sqrt{17}$

Circle centre = mid-point of AB

$$= (-3, 4)$$

So, the equation of the circle is $(x+3)^2 + (y-4)^2 = 17$

2016 Paper 1 Question 8, (5)

Substituting $y = 3x - 5$ into the circle equation gives

$$x^2 + (3x - 5)^2 + 2x - 4\left(3x - 5\right) - 4 = 0$$

$$x^2 + 9x^2 - 30x + 25 + 2x - 12x + 20 - 4 = 0$$

$$10x^2 - 40x + 40 = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x = 2$$

Since there is only one point of intersection the line must be a tangent to the circle

When $x = 2, y = 1$

So, the point of tangency is $(2, 1)$

2016 Paper 2 Question 4, (4) (3)

a) For C_1 centre $= (-5, 6)$ radius $= 3$

For C_2 centre $= (3, 0)$ radius $= 5$

b) Distance between the centre of C_1 $C_2 = \sqrt{(6 - 0)^2 + (-5 - 3)^2}$

$$= \sqrt{36 + 64}$$

$$= 10$$

Since the sum of the radii $= 3 + 5 = 8 < 10$ the circles cannot intersect

2017 Paper 1 Question 2, (4)

$$x^2 + y^2 - 8x - 6y - 15 = 0$$

Centre $= (4, 3)$

Let $C = (4, 3)$

$m_{CP} = \dfrac{3-1}{4-(-2)} = \dfrac{1}{3}$

$m_{tan} = -3$ since perpendicular to CP

Using $y - b = m(x - a)$ with $(-2, 1)$ gives

$y - 1 = -3(x - (-2))$

$y = -3x - 5$

<u>2017 Paper 2 Question 3, (5)</u>

$(x-2)^2 + (y-1)^2 = 25$

Substituting $y = 3x$ into the circle equation gives

$(x-2)^2 + (3x-1)^2 = 25$

Expanding brackets and simplifying gives

$x^2 - 2x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1, x = 2$

When $x = -1, y = -3$ giving $(-1, -3)$

When $x = 2, y = 6$ giving $(2, 6)$

<u>2017 Paper 2 Question 10, (3) (4)</u>

a) $A = (-7, -2),\ B = (2, 1),\ C = (17, 6)$

$$\vec{AB} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

So, $\overrightarrow{AB} = \frac{3}{5}\overrightarrow{BC}$ showing that $\overrightarrow{AB}$ $\overrightarrow{BC}$ are parallel and since B is a common point A, B, & C are collinear

b) $r_A = \sqrt{10}$

$r_B = 2\sqrt{10}$

$r_C = r_A + r_B = 3\sqrt{10}$

By inspection of the graph the radius of the circle with centre D is $6\sqrt{10}$

The point D divides the line AC in the ratio $5:3$

So, $\overrightarrow{AD} = \frac{5}{8}\overrightarrow{AC} = \frac{5}{8}\begin{pmatrix} 24 \\ 8 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$

Then $D = \left(-7 + 15,\ -2 + 5\right) = (8, 3)$

So, the circle centre is (8, 3)

$(x - 8)^2 + (y - 3)^2 = (6\sqrt{10})^2$

$(x - 8)^2 + (y - 3)^2 = 360$